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THE CALCULATION OF STREAMLINES IN AN INHOMOGENEOUS, ANISOTROPIC POROUS MEDIUM WITH TWO-DIMENSIONAL FLOW

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INTRODUCTION

With respect to quality problems of groundwater it is important to know the streamlines and traveltimes in a flow region. Especially in flow problems in the vertical plane e.g. underneath a waste disposal, the presence of inhomogeneities, such as clay layers, is very important.
In this paper a method will be presented for the calculation of the streamfunction all through the flow domain based upon the finite element method. In this flow domain which is divided in elements with various transmissivities internal sources or sinks may be present. Also the porous medium can be anisotropic.

THE BASIC EQUATIONS

In twodimensional groundwaterflow Darcy's law states

$$\left.\begin{aligned} q_x &= -k_{xx}\frac{\partial\varphi}{\partial x} - k_{xy}\frac{\partial\varphi}{\partial y} \\ q_y &= -k_{xy}\frac{\partial\varphi}{\partial x} - k_{yy}\frac{\partial\varphi}{\partial y} \end{aligned}\right\} \qquad (1)$$

where q_x and q_y are the components of the specific discharge in the x and y direction of a cartesian coördinate system, is the groundwaterhead and k is the permeability.
With the continuity equation

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \qquad (2)$$

this results in the Laplace equation which can be solved in a flow domain with known boundary conditions with help of the finite element method.

However there can be stated that a streamfunction ψ exists such that

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$$\left.\begin{aligned} q_x &= -\frac{\partial \psi}{\partial y} \\ q_y &= \frac{\partial \psi}{\partial x} \end{aligned}\right\} \tag{3}$$

From 1 and 3 it follows

$$\left(k_{xx}k_{yy}-k_{xy}^2\right)\frac{\partial \varphi}{\partial y} = -k_{xy}\frac{\partial \psi}{\partial y} - k_{xx}\frac{\partial \psi}{\partial x}$$

$$\frac{\partial \varphi}{\partial y} = -\frac{k_{xy}}{k_{xx}k_{yy}-k_{xy}^2}\frac{\partial \psi}{\partial y} - \frac{k_{xx}}{k_{xx}k_{yy}-k_{xy}^2}\frac{\partial \psi}{\partial x} \tag{4}$$

$$\frac{\partial \varphi}{\partial x} = \frac{k_{xy}}{k_{xx}k_{yy}-k_{xy}^2}\frac{\partial \psi}{\partial x} + \frac{k_{yy}}{k_{xx}k_{yy}-k_{xy}^2}\frac{\partial \psi}{\partial y} \tag{5}$$

$$\frac{\partial}{\partial y}\left(-\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) = 0 \tag{6}$$

and equation 4 and 5 and if the permeabilities in the main directions are k_x and k_y it follows:

$$\begin{aligned} &\frac{\partial}{\partial y}\left(-\frac{k_{xy}}{k_x k_y}\cdot\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{k_{yy}}{k_x k_y}\cdot\frac{\partial \psi}{\partial y}\right) + \\ &+ \frac{\partial}{\partial x}\left(-\frac{k_{xy}}{k_x k_y}\cdot\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial x}\left(-\frac{k_{xx}}{k_x k_y}\cdot\frac{\partial \psi}{\partial x}\right) = 0 \end{aligned} \tag{7}$$

If the main directions of the permeabilities coincide with the coördinates x and y of the cartesian coördinate system equation 7 reduces to

$$\frac{\partial}{\partial y}\left(-\frac{1}{k_x}\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial x}\left(-\frac{1}{k_y}\frac{\partial \psi}{\partial x}\right) = 0 \tag{8}$$

(see Bear "Dynamics of fluids in porous media").

Equation 8 can be solved numerically by means of the finite element method. Along the flow domain boundary the boundary conditions have to be known in terms of the streamfunction.

SINKS AND SOURCES IN THE FLOW DOMAIN

In the case of the steady radial flow towards a well in a homogeneous porous medium the complex potential is

$$\Omega = \frac{Q}{2\pi} \ln z + A \tag{9}$$

where Q is the discharge of the well and A a constant value depending on the boundary condition of the flow domain and the choice of the zero level of the potential and the streamfunction. This means for the streamfunction

$$\psi = \frac{Q}{2\pi} \theta + I_m(A) \tag{10}$$

and this streamfunction appears to be many-valued. Generally it can be deduced that

$$\psi = \frac{Q}{2\pi} \theta + \psi^*$$

where ψ^* is a single valued function. One can state without any loss of generality that the angle θ is chosen in the interval

$$0 \leqslant \theta < 2\pi$$

The streamfunction is now single valued where ψ^* is a constant value.

If there is a sink or source situated in a nodal point of an element mesh with different permeabilities in the surrounding elements as indicated in figure 1 the assumption is made that in the vicinity of the well there is still radial symmetry with respect to the groundwaterheads. This means that the flow is still radial and that

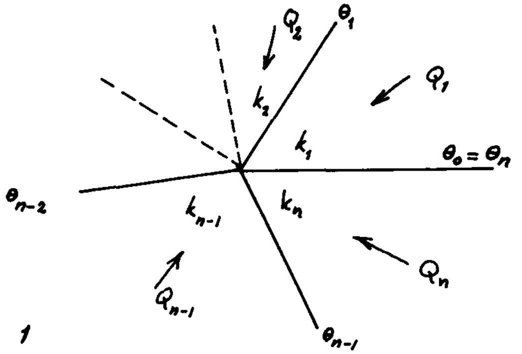

fig. 1

$$\varphi = \frac{Q_i}{(\theta_i - \theta_{i-1})k_i} \ln r$$

$$Q_i = \frac{\varphi}{\ln r} (\theta_i - \theta_{i-1}) k_i \qquad (12)$$

Since the total discharge of the well

$$Q_T = \sum_{i=1}^{n} Q_i \qquad (13)$$

it follows that

$$Q_T = \frac{\varphi}{\ln r} \sum_{i=1}^{n} (\theta_i - \theta_{i-1}) k_i \qquad (14)$$

From equation 12 it follows

$$Q_i = a (\theta_i - \theta_{i-1}) k_i \qquad (15)$$

where $a = \frac{\varphi}{\ln r}$

and with equation 14

$$a = \frac{Q_T}{\sum_{i=1}^{n} (\theta_i - \theta_{i-1}) k_i} \qquad (17)$$

Now it can be stated that the streamfunction around the well in a point p with an angle θ is

$$\psi_p = \psi^* + \sum_{j=1}^{i} Q_j + \frac{\theta - \theta_i}{\theta_{i+1} - \theta_i} Q_{i+1} \qquad (18)$$

The use of equation 18 in the finite element method means that the value of the streamfunction which has to be calculated in the nodal point with a well can be separated into two parts. The first part is the unknown but constant part ψ^* and the second is the part depending on the value θ. This latter one however can directly be solved from the configuration of the elements and from the permeabilities and the strength of the discharge Q_T. From figure 1 can be seen that the discontinuity at $\theta = 0$ in the streamfunction has to be pursued along the sides of elements until it reaches the boundary. This means along this element sides the application of a line doublet. This line doublet causes the jump in the ψ over the element sides. On both sides of the line doublet the streamfunction has a different value and these values are

$$\left.\begin{aligned} \psi &= \psi^{*} \\ \psi &= \psi^{*} + Q_{T} \end{aligned}\right\} \qquad (19)$$

As an example the method is applied in a simple finite element program using squares as elements.

EXAMPLE

A waste disposal is situated on the land surface. There is a percolation of water out of the disposal towards the groundwater of 0.2 m/day. As indicated in figure 1. the upper aquifer has a thickness of 20 m. The lower aquifer has a thickness of 60 m. and they are separated by a semi-pervious layer with a hydraulic resistance of 100 days. There is assumed that there is only flow in the vertical plane. An extraction in the lower aquifer is schematized to a drain at a height of 30 m. above the impervious base. The width of the disposal is 20 m. In order to calculate the streamfunction all through the flowdomain as schematised in figure 2. the flowdomain is divided in square elements. In figure 3. this 126 elements and 150 nodal points are indicated. With the boundary conditions as indicated in figure 2. the calculation is done. The theory of the former chapters is applied in the nodal points at the waste disposal and in the drain. Also along the arbitrarely chosen line doublet this is done. The application of the finite element method delivers a matrix from which the streamfunction in every nodal point is calculated. At the line doublet the jump into the streamfunction is realised. With interpolation and the interpretation of the jump along the line doublet the streamlines are as indicated in figure 3.

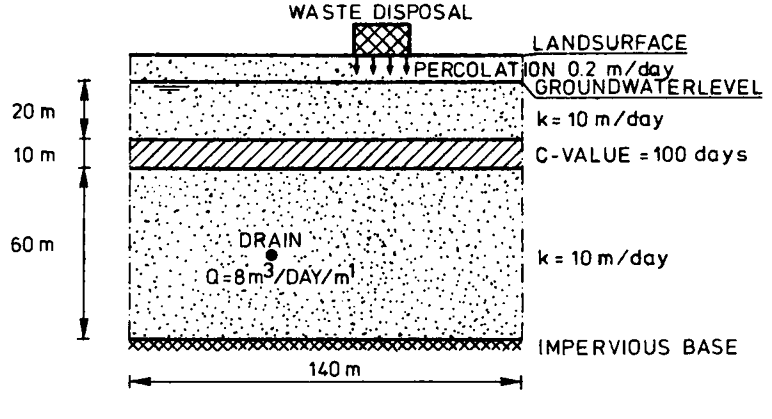

FIG. 1

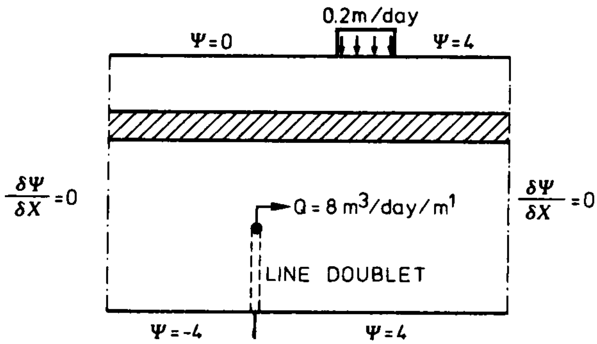
0.2m/day
Ψ=0
Ψ=4
δΨ/δX =0
δΨ/δX =0
Q = 8 m3/day/m1
LINE DOUBLET
Ψ=-4
Ψ=4

FIG. 2

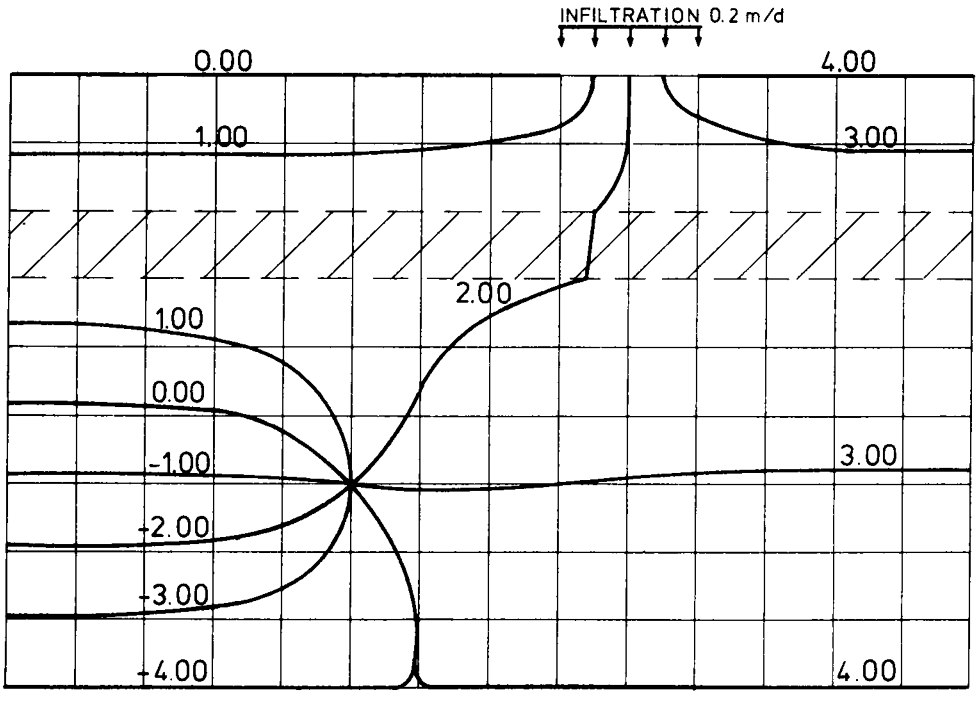
INFILTRATION 0.2 m/d
0.00
4.00
1.00
3.00
2.00
1.00
0.00
-1.00
3.00
-2.00
-3.00
-4.00
4.00

FIG. 3