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CALCULATION OF FLOWPATTERNS AND TRAVELLING TIMES IN GROUNDWATERFLOW WITH RECTANGULAR HERMITIAN ELEMENTS.

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ABSTRACT

One of the first tasks in modeling groundwater quality problems is the simulation of the groundwater flow regime in an adequate way. The simulation of transport of contaminants requires an accurate description of the velocity field.

Rectangular finite elements with first order continuous Hermitian basis functions can be used for the calculation of potentials and velocities in two dimensional steady groundwater flow including inhomogeneities and anisotropy. However, the resulting approximation is not accurate enough to describe the behaviour of the groundwater potential in the vicinity of point sources and sinks. A much better approximation in these regions is obtained by extending the interpolation function for the groundwater potential with a basis function which shows a logarithmic behaviour in the vicinity of the wells.

Results of this approach show a numerically calculated groundwater potential and velocity in the neighbourhood of the wells which are very close to the exact solution, even for distances from the wells which are small compared to the elementsize.

INTRODUCTION

Convective transport of contaminatns in groundwater in general plays an important role in groundwater quality problems. As a consequence, the transport of contaminants in groundwater can only be simulated in an adequate way if the groundwater flow regime can be described with an acceptable degree of accuracy.

The finite element method can be a very useful method for the solution of the equations that describe the hydrological system and hence for the calculation of the groundwater flow field. In order to obtain a continuous groundwater velocity over the domain considered, higher order elements have to be used. In the approach described in this paper, these elements are rectangular with first order continuous Hermitian basis functions.

Problems, however, will occur in the vicinity of point sources and sinks, where the groundwater potential varies strongly over short distances. In general, the approxima-

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tion to the groundwater potential in these regions is not accurate enough, unless a very large number of small elements is used. This paper describes an attempt to solve this problem in an approximate way. For each well a function, containing one unknown coefficient, is defined to describe the groundwater potential in the vicinity of the well. In the finite element formulation of the problem, these well functions are treated as basis functions, and the unknown coefficients are calculated.

THEORETICAL APPROACH

Let us consider steady groundwater flow in a semi-confined or confined aquifer. Treating the aquifer as two-dimensional horizontal with transmissivity T, the governing equation can be written as:

$$\frac{\partial}{\partial x}\{T_{xx}\frac{\partial h}{\partial x}\} + \frac{\partial}{\partial y}\{T_{yy}\frac{\partial h}{\partial y}\} + q = 0 \tag{1}$$

where:

T_{xx}, T_{yy} = transmissivity in x and y direction (m^2/d)
h = groundwater potential (m)
q = recharge (m/d)

It has been assumed that the principal axes of the transmissivity tensor coïncide with the x and y directions.

In most practical cases an approximate method has to be used to solve equation (1) for the groundwater potential h. Whenever the solution to equation (1) is known, velocities in x and y directions can be calculated from Darcy's law:

$$v_x = -\frac{T_{xx}}{n_e d}\frac{\partial h}{\partial x}$$
$$v_y = -\frac{T_{yy}}{n_e d}\frac{\partial h}{\partial y} \tag{2}$$

where:

v_y, v_y = velocities in x and y direction (m/d)
n_e = effective porosity
d = aquifer thickness (m)

The Galerkin finite element method is a very useful method to solve equation (1) in an approximate way. The application of this method will not be discussed here in detail (ref. 1). The point of interest is that an approximate solution has to be defined in the form:

$$\bar{h} = \sum_{j=1}^{N} C_j \phi_j(x,y) \tag{3}$$

where:

C_j = unknown coefficients N = number of basis functions
ϕ_j = known basis functions

The unknown coefficients C_j can than be calculated and the approximate solution is known in every point of the domain considered.

BASIS FUNCTIONS

Very often the basis functions ϕ_j are taken to vary linear or quadratic over the elements of the finite element network, such that $\bar{h}$ is continuous over the domain considered and the coefficients C_j are the values of the approximate solution in the nodal points. In that case the derivatives of $\bar{h}$ with respect to x and y will be continuous in an element, but will show a jump across the interelement boundaries. In order to obtain a continuous velocity field, one can either use a smoothing or averaging technique (e.g. ref. 2, 3) or turn to higher order basis functions which will result in continuous derivatives of $\bar{h}$ at the interelement boundaries (e.g. ref. 4). The latter approach has been chosen here by applying rectangular elements with first order continuous Hermitian basis functions. The unknown coefficients C_j are in this case given by h, $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial y}$ and $\frac{\partial^2 h}{\partial x \partial y}$ at the nodal points (ref. 5).

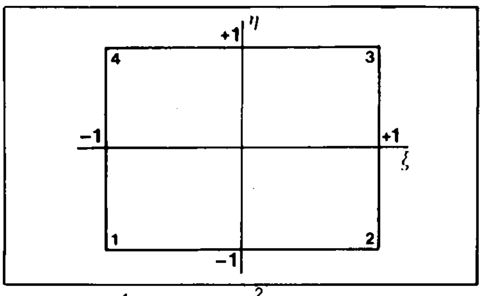


Fig. 1 Elementshape in local coordinates.

In a local coordinate system (fig. 1) the two-dimensional first order continuous Hermitian basis functions are given by (ref. 6):

$$
\begin{aligned}
H_{00} &= \frac{1}{16} (\xi + \xi_0)^2 (\xi\xi_0 - 2) (\eta + \eta_0)^2 (\eta\eta_0 - 2) \\
H_{10} &= -\frac{1}{16} \xi_0 (\xi + \xi_0)^2 (\xi\xi_0 - 1)(\eta + \eta_0)^2 (\eta\eta_0 - 2) \\
H_{01} &= -\frac{1}{16} (\xi + \xi_0)^2 (\xi\xi_0 - 2) \eta_0 (\eta + \eta_0)^2 (\eta\eta_0 - 1) \\
H_{11} &= \frac{1}{16} \xi_0 (\xi + \xi_0)^2 (\xi\xi_0 - 1) \eta_0 (\eta + \eta_0)^2 (\eta\eta_0 - 1) \\
\xi_0, \eta_0 &= \pm 1
\end{aligned}
\tag{4}
$$

The interpolation function for the approximate solution is given by:

$$
\bar{h} = \sum_{j=1}^{4} \left(\bar{h}_j H_{00_j} + \frac{\partial \bar{h}_j}{\partial \xi} H_{10_j} + \frac{\partial \bar{h}_j}{\partial \eta} H_{01_j} + \frac{\partial^2 \bar{h}_j}{\partial \xi \partial \eta} H_{11_j}\right) \tag{5}
$$

If the element in the global coordinate system is rectangular, it can be shown (ref.5) that this interpolation function leads to continuous normal and tangential derivatives of $\bar{h}$ at the interelement boundaries and hence to a continuous velocity field, provided the transmissivities (T_{xx}, T_{yy}), effective porosity (n_e) and the aquiferthickness (d) are continuous.

The transformation from local to global coordinates is a simple linear one, and consequenly the groundwater potential in each element is approximated by a bi-cubic polynomial in x and y.

WELL FUNCTION

In the vicinity of point sources and sinks, where the groundwater potential varies strongly over short distances, a bi-cubic polynomial approximation is in general not accurate enough, unless a very large number of small elements is used. To overcome this problem one can either use an analytical solution around the well and couple this solution to the finite element solution in the remaining part of the domain (ref. 2, 3) or extend the interpolation function (equation 5) with a basis function that describes the behaviour of $\bar{h}$ in the vicinity of the wells (ref. 7, 8). The latter approach has been chosen here.

A well function F_w is defined for a circular region with radius r_m around each well and the interpolation function for $\bar{h}$ in an element containing a well is written as:

$$\bar{h} = \sum_{j=1}^{4} \left(\bar{h}_j H00_j + \frac{\partial \bar{h}_j}{\partial \xi} H10_j + \frac{\partial \bar{h}_j}{\partial \eta} H01_j + \frac{\partial^2 \bar{h}}{\partial \xi \partial \eta} H11_j\right) + c.F_w \qquad (6)$$

where c is an unknown coefficient which has to be calculated. For reasons of continuity F_w has to fulfil two requirements:

$F_w = 0$ at $r = r_m$, which means that $\bar{h}$ is continuous at $r = r_m$

$\frac{\partial F_w}{\partial x} = 0$ and $\frac{\partial F_w}{\partial y} = 0$ at $r = r_m$, which means that both $\frac{\partial \bar{h}}{\partial x}$ and $\frac{\partial \bar{h}}{\partial y}$ are continuous at $r = r_m$

F_w can tentatively be defined by:

$$F_w = \tfrac{1}{2}\left(1 - \frac{r_w^2}{r_m^2}\right) + \ln\left(\frac{r_w}{r_m}\right) \qquad r < r_w$$

$$= \tfrac{1}{2}\left(1 - \frac{r^2}{r_m^2}\right) + \ln\left(\frac{r}{r_m}\right) \qquad r_w < r < r_m \qquad (7)$$

$$= 0 \qquad r > r_m$$

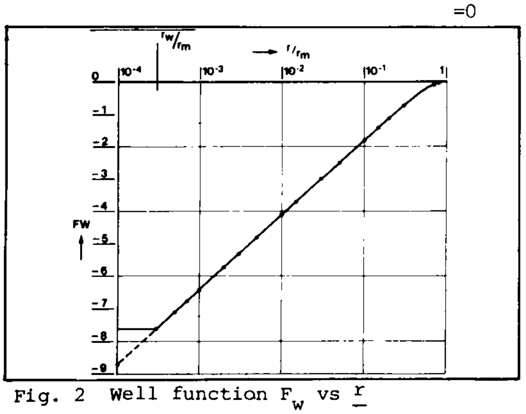


Fig. 2 Well function F_w vs $\frac{r}{r_m}$

The shape of this well function is given in fig. 2.

In order to limit the amount of work that has to be done, the region over which F_w is nonzero (hence the choice of r_m) should be such that this region lies completely within one element.

EXAMPLES

Two examples are given here to compare numerical results with existing analytical solutions. In the first example groundwater is abstracted from a semi-confined homogeneous, isotropic aquifer through a single well. The model area is square (10 000 x 10 000 m^2) and together with the position of the nodal points and the abstraction point is shown in fig. 3.

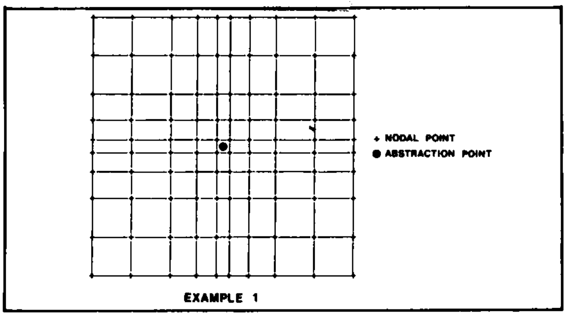


Fig. 3. Modelarea for example 1.

The relevant properties for this example are as follows:

transmissivity T = 1000 m^2/d, aquifer thickness d = 100 m,
effective porosity n_e = 0.25, confining layer $\frac{d'}{k'}$ = 1000 d,
abstraction rate Q = 3000 m^3/d and well radius r_w = 0.15 m

A fixed potential is maintained on the boundary. For these properties the areal extent of the aquifer can be considered infinite and the numerical results can be compared to a simple analytical solution (ref. 9). Fig. 4 shows a comparison of the analytical and numerical calculated velocities towards the well.

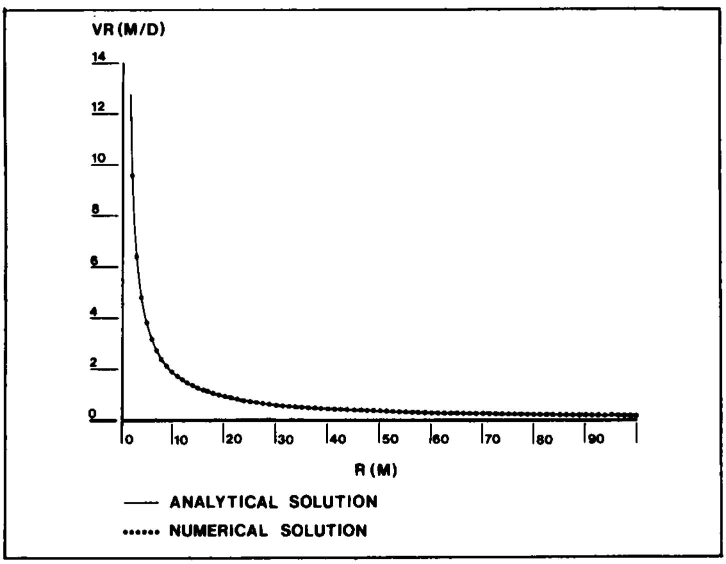
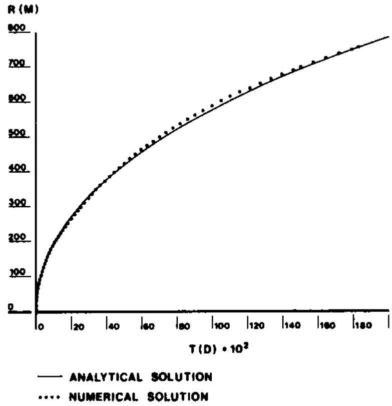


Fig. 4 Velocity of a waterparticle towards the well vs. distance from the well

The solutions compare very well, even for distances from the well as small as 2 m. Residence times, i.e. the time it takes a water particle at distance r from the well to reach the well, have also been calculated. Fig. 5 shows a comparison of the analytical and numerical results.

Fig. 5
Residence times vs.
distance from the well



In example 2 groundwater is abstracted from a homogeneous, isotropic aquifer through two wells. The model area is square (10 000 x 10 000 m^2) and is shown, together with the position of the nodal points and the abstraction points in fig. 6.

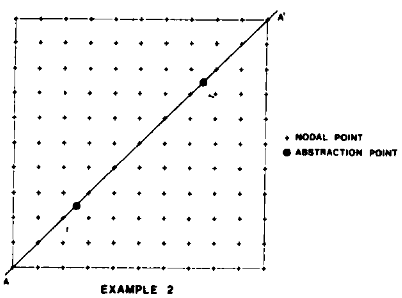


Fig. 6. Model area for example 2

The aquifer is recharged at a constant rate of 0.0002 m/d. Abstraction rates are Q_1 = 8 000m^3/d and Q_2 = 12 000 m^3/d. The aquifer properties and boundary conditions are the same as for example 1. An analytical solution can be formulated for this problem. Fig. 7 shows a comparison of the analytical and numerical solution for the groundwater potentical along cross-section A-A'.

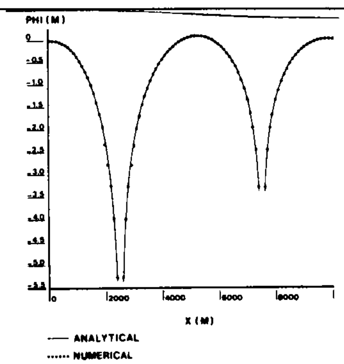


Fig. 7. Groundwater potential along cross-section A-A'

Fig. 8 and 9 give the same comparison along the same cross-section for a small region around the wells.

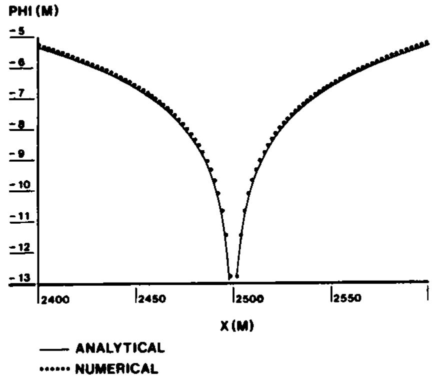
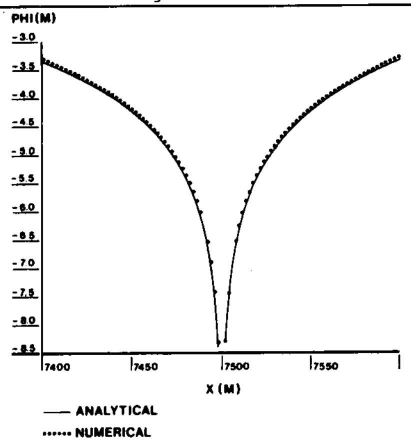


Fig. 8 Groundwater potential along cross-section A-A'

Fig. 9. Groundwater potential along cross-section A-A'



Clearly, the analytical and numerical results compare very well, even for distances from the wells as small as 3 m, bearing in mind that the element size is 1 000 x 1 000 m^2.

CONCLUSION

The two examples given show, that the groundwater potential and velocity in two-dimensional steady groundwater flow can very well be approximated by using rectangular finite elements with first order continuous Hermitian basis functions in combination with a well function to describe the behaviour of the groundwater potential in the vicinity of a well. Results compare very well with analytical solutions, even for distances from the wells which are very small compared to the element size.

Due to the nature of the well function, the amount of work that has to be done to evaluate integrals arising from the Galerkin formulation of the problem, increases. This increase is in general not too serious because the number of wells is usually limited.

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