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OBSERVATION OF AQUIFER POLLUTION STATE

M.NAWALANY
Institute of Environmental Engineering,
Warsaw Technical University, Warsaw (Poland)

ABSTRACT

Basic equations that describe aquifer pollution processes have been presented in the paper in a form of state equations. Such a form enables one to check whether the processes are observable providing the given set of discrete observations is available. The simple criterion of observability has also been given. The main idea of the technique proposed is to answer objectively - as far as that is possible - one of the most important engineering questions of groundwater pollution: is the aquifer pollution state observable from a given observation network ?

INTRODUCTION

Mathematical description of the state of aquifer pollution consists of two partial differential equations - the Bussinesq's equation and the hydrodynamic dispersion equation. Analytical solutions of the equations are available only for very few cases of extremely simple aquifer geometries. To cope with dimensionality, complicated aquifer geometries, spatial distribution of sources and physical properties of aquifer various numerical techniques of space discretisation have been developed. The main idea of such discretisation is to transform partial differential equations that describe the original problem into so called state equations. The state equations description of the processes enables one to perform both qualitative and quantitative analyses of aquifer pollution. In the following sections one qualitative method is described in detail, namely the observability test of aquifer pollution state.

THE STATE EQUATIONS

To solve properly any aquifer pollution problem one has to describe mathematically all physical and chemical processes that occur in the aquifer under consideration. Usually two well known equations are used:

- the Boussinesq's equation together with its initial and boundary conditions (ref. 1,2)

$$\mu \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}(T \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(T \frac{\partial h}{\partial y}) + S_h , \quad h(x,y,0) = h_o(x,y) \tag{1a}$$

appropriate boundary conditions

where h - piezometric head,
T - transmissivity,
μ - storage coefficient,
S_h - source term,

- the hydrodynamic dispersion equation together with its initial and boundary conditions (ref. 1,3)

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}(D \frac{\partial C}{\partial x}) + \frac{\partial}{\partial y}(D \frac{\partial C}{\partial y}) - \frac{\partial}{\partial x}(v_x C) - \frac{\partial}{\partial y}(v_y C) + S_c , \tag{1b}$$

$$C(x,y,0) = C_o(x,y)$$

appropriate boundary conditions

where C - concentration of pollutant,
D - dispersion coefficient,
$\underline{v}$ - specific discharge,
S_c - source term,

The mathematical form of the equations (1a) and (1b) is not very usefull for engineering practice unless the geometry of the aquifer is simple and the physical characteristics of the processes are constant in time. To copy with the complexity of the mathematical description of aquifer processes one has to use a discretisation technique (for example the finite element method) to transform partial differential equations into a system of ordinary differential equations. The resulting system - called state equation - is a simplification that enables one to express the answer to the original question in a numerical (quantitative) way. The state equations (1a) and (1b) have usually the following form:

$$D_1\dot{\underline{h}} = A_1\underline{h} + B_1\underline{u}_1 , \qquad \underline{h}(0) = \underline{h}_o , \tag{2a}$$

$$D_2\dot{\underline{C}} = A_2\underline{C} + B_2\underline{u}_2 , \qquad \underline{C}(0) = \underline{C}_o . \tag{2b}$$

where $\underline{h},\underline{C}$ - state vectors (piezometric head and pollutant concentration in a series of discrete points of aquifer),

$D_i\,(i=1,2)$ - geometry matrices,

$A_i\,(i=1,2)$ - physical characteristics and boundary conditions matrices,

$B_i\,(i=1,2)$ - sources distribution matrices,

$\underline{u}_i\,(i=1,2)$ - source vectors.

The explicit relationships between the original equations parameters and the elements of A_i and D_i matrices have been given by many authors (ref. 2,3) and will not be repeated here. The forms (2a) and (2b) of the state equations are suitable for purposes of simulation or parameter identification. In such cases the equations are solved one after another by the appropriate numerical integration scheme (e.g. the Crank - Nicholson scheme). The important features of the state equations matrices are as follows: in many interesting cases D_i matrices are positive definite, while A_i matrices are symmetrical. Positive definition of D_i matrices can immediatelly be derive from the Gershgorin's theorem (ref. 4). Starting from this observation one can transform equations (2a) and (2b) into a form which is more useful in qualitative analysis, namely

$$\dot{\underline{h}} = \tilde{A}_1\underline{h} + \tilde{B}_1\underline{u}_1 , \qquad \underline{h}(0) = \underline{h}_o , \tag{3a}$$

$$\dot{\underline{C}} = \tilde{A}_2\underline{C} + \tilde{B}_2\underline{u}_2 , \qquad \underline{C}(0) = \underline{C}_o . \tag{3b}$$

where $\tilde{A}_i = D_i^{-1}A_i , \quad (i = 1,2),$

$\tilde{B}_i = D_i^{-1}B_i , \quad (i = 1,2).$

Finally the state equations can be transformed into the canonical form in which A_i matrices become diagonal:

$$\dot{\underline{\xi}}_1 = \Lambda_1 \underline{\xi}_1 + \tilde{\tilde{B}}_1 \underline{u}_1 , \qquad \underline{\xi}_1(0) = \underline{\xi}_{10} , \tag{4a}$$

$$\dot{\underline{\xi}}_2 = \Lambda_2 \underline{\xi}_2 + \tilde{\tilde{B}}_2 \underline{u}_2 , \qquad \underline{\xi}_2(0) = \underline{\xi}_{20} , \tag{4b}$$

where $\Lambda_i = V_i^{-1} A_i V_i$ - diagonal matrices similar to A_i , $(i = 1,2)$,

V_i - orthogonalisation matrices, $(i = 1,2)$,

$\underline{\xi}_1 = V_i^{-1} \underline{h}$ - canonical form of the state vector $\underline{h}$,

$\underline{\xi}_2 = V_2^{-1} \underline{C}$ - canonical form of the state vector $\underline{C}$,

$\tilde{\tilde{B}}_i = V_i^{-1} \tilde{B}_i$ - transformed $\tilde{B}_i$ matrix, $(i = 1,2)$.

With the state equations in the canonical form one can proceed to the observability question.

OBSERVABILITY OF THE AQUIFER PROCESSES.

To formulate any statement on aquifer pollution every engineer begins from observation of the piezometric head and measurement of pollutant concentrations at various discrete points of the aquifer. The measuring procedure can be described mathematically by the observation equations:

$$\underline{z}_1 = H_1 \underline{h} + \underline{v}_1 \tag{5a}$$

$$\underline{z}_2 = H_2 \underline{C} + \underline{v}_2 \tag{5b}$$

where H_i $(i = 1,2)$ - observation matrix,

$\underline{z}_i$ $(i = 1,2)$ - observation vector,

$\underline{v}_i$ $(i = 1,2)$ - observation disturbances vector

or in the canonical form

$$\underline{z}_1 = \tilde{H}_1 \underline{\xi}_1 + \underline{v}_1 \tag{6a}$$

$$\underline{z}_2 = \tilde{H}_2 \underline{\xi}_2 + \underline{v}_2 \tag{6b}$$

where $\tilde{H}_i = H_i V_i$, $(i = 1,2)$ - canonical form of observation matrices.

The observation matrices $\tilde{H}_i$ play or rather should play an important role in any measuring procedure for two reasons:

- $\tilde{H}_i$ contains information about the system under consideration. The information is incorporated into the canonical transformation matrix V_i which consists of eigenvectors of the system matrix A_i.
- $\tilde{H}_i$, being a transformed matrix H_i, contains also direct information on positions of the measuring points.

Now, the following question arises: can we estimate aquifer state vectors $\underline{h}$ and $\underline{C}$ by measuring piezometric head and pollutant concentration only in a given set of observation points ? The question is known as the observability problem. Till now very little has been said about the concept of observability in the context of aquifer pollution. In the author's opinion the answer to the question is a crucial point for any engineering and scientific activity in aquifer pollution problems. The following criterion (ref. 5,6) can easily be checked in order to solve the observation question: "A system is totally observable if and only if the $\tilde{H}_i$ matrices do not contain any zero-column". The necessity of the condition mentioned above is evident while the proof of sufficiency needs some matrix operations and will not be repeated here. To analyse $\tilde{H}_i$ matrices with the above criterion it is necessary to know V_i matrices (i.e. the state equations) and H_i matrices (i.e. the potential or real positions of observation points). Finally it should be mentioned that in practice the $\tilde{H}_i$ matrices that arise from the continuous process equations (1a) and (1b) never have a zero-column, unless the space discretisation of an aquifer is very peculiar. What one finds in many practical cases is that one or more columns of $\tilde{H}_i$ matrices are nearly zero-column. It means that state vectors are in practice unobservable. Such a result of the qualitative analysis of aquifer pollution processes implies a change in the existing or planned observation network.

RESULTS AND CONCLUSIONS

From the above considerations the following algorithm of observation test can be recommended:

- starting from a priori knowledge of a given aquifer write the state equations for pollution processes,

- transform the state equations into canonical form and remember the transformation matrices V_i $(i = 1,2)$,
- write the observation equations (matrices H_i) for a given set of observation points,
- form the canonical observation matrices $\tilde{H}_i$ $(i = 1,2)$,
- analyse the columns of $\tilde{H}_i$ matrices: if $\tilde{H}_i$ $(i = 1,2)$ has no zero column the system under consideration is observable, if not change the observation network.

It is therefore clear that the observability test can be helpful in two important activities:

- as a basic tool for designing observation networks, and
- as a indispensible step in the prediction of pollutant concentration in the aquifer (concentration of pollutant in unobservable points cannot be predicted).

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