Quality of Groundwater, Proceedings of an International Symposium, Noordwijkerhout, The Netherlands, 23–27 March 1981, W. van Duijvenbooden, P. Glasbergen and H. van Lelyveld (Eds.), *Studies in Environmental Science*, Volume 17

STOCHASTICAL ANALYSIS OF GROUNDWATER FLOW TOWARDS A WELL IN A HETEROGENEOUS AQUIFER

A.N.M. OBDAM

National Institute for Water Supply, Voorburg, The Netherlands

ABSTRACT

Aquifers are essentially inhomogeneous due to the way they were formed. For instance river depositions are built up out of sandbars and sandripples which may be intercalated with silt or clay layers. For that reason the main parameter of groundwater flow, the hydraulic permeability is in general not correctly described by its mean value but should be characterized by two stochastical functions. The first is the frequency distribution, independent of the place in the aquifer and the second is the autocorrelation function in space. These stochastical properties of the permeability may have an important influence on the characteristics of groundwater flow through an aquifer.

For the main characteristics of groundwater flow sets of stochastic differential equations are derived containing their auto- and crosscorrelation functions as variables.

INTRODUCTION

Deterministic models are commonly used for the calculation of groundwater flow. One of the assumptions for these calculations is, that the hydraulic properties of the geologic formation in which the flow takes place can be regarded as constant over some distance.

The validity of this assumption in case of an un-consolidated sedimentary structure is influenced in two respects by the geologic way the aquifer was formed: horizontal layering consisting of sandripples, cross beddings and larger structures and the finite horizontal extension of these structures (ref. 1). This paper only deals with the last aspect. Regarding the manner in which a sedementary aquifer was formed one may describe the permeability in a stochastical way. This stochastical description uses two functions: First the frequency distribution expressing for a range of values the chance finding, by means of a field test, a specific small range of values (not taking into account measuring errors!) and second the autocorrelation function in space. This is a function of the correlation coefficient of permeability values, found in field tests, with the distance between the test sites. Concerning the frequency distribution of the permeability, many indications in literature can be found (ref. 2 and 3) that the logarithm of the permeability has a Gauss frequency distribution (is normally distributed). According to Whittle (ref. 4) for a two dimensional autocorrelation function, preference should be given to $b\xi\, K_1(b\xi)$ in which b is a constant to be determined in the field, ξ is the relative distance in the field and $K_1(--)$ is a modified Bessel function of the second kind, of the first order.

In the following a set of differential equations will be derived, with the auto- and crosscorrelation functions of the drawdown of the groundwater and the logarithm ln(k) of the permeability as variables for a particular case of groundwater flow: two dimensional horizontal flow of groundwater towards a well in a confined aquifer with replenishment at a fair distance R from the well. This aquifer is regarded to be physically and statistically isotropic and homogeneous. For this reason points of equal

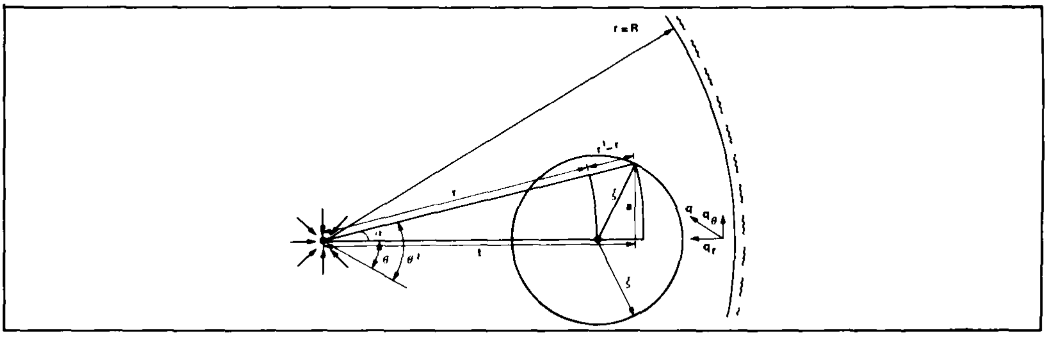

Fig. 1. Circular set of points of equal autocorrelation of the permeability to a certain point in a circular aquifer.

autocorrelation to a certain chosen point form a circle, as is shown in fig. 1. The relation in this figure between ξ, r, r' and α is the following, according to the cosine rule: $r'^2 + r^2 - 2rr'\cos\alpha=\xi^2$.

DERIVATION OF THE AUTOCORRELATION FUNCTION OF THE DRAWDOWN OF THE GROUNDWATER.

The well flow takes place in a medium of which the permeability is regarded to be a stochastic variable, with an areal autocorrelation function. It can consequently be expected, that the well flow will have a statistical variation around its mean conduct. The stochastic properties of the characteristics of the well flow can be expressed in their auto- and crosscorrelation functions. For the drawdown of the groundwater ϕ the following derivation can be made: Continuity equation and Darcy's Law give:

$$\frac{\delta^2\phi}{\delta r^2} + \frac{1}{k}\,\frac{\delta k}{\delta r}\,\frac{\delta\phi}{\delta r} + \frac{1}{r}\,\frac{\delta\phi}{\delta r} + \frac{1}{r^2}\left(\frac{\delta^2\phi}{\delta\theta^2} + \frac{1}{k}\,\frac{\delta k}{\delta\theta}\,\frac{\delta\phi}{\delta\theta}\right) = 0 \qquad (1)$$

In which:

ϕ = drawdown of the groundwater {L}

k = horizontal permeability {LT^{-1}}

r = radial distance from the well {L}

θ = angular coordinate {-}

Using $\frac{1}{k}\frac{\delta k}{\delta x} = \frac{\delta \ln k}{\delta x}$ (1) becomes, with f = ln(k)

$$\frac{\delta^2\phi}{\delta r^2} + \frac{\delta f}{\delta r}\frac{\delta\phi}{\delta r} + \frac{1}{r}\frac{\delta\phi}{\delta r} + \frac{1}{r^2}\left(\frac{\delta^2\phi}{\delta\theta^2} + \frac{\delta f}{\delta\theta}\frac{\delta\phi}{\delta\theta}\right) = 0 \qquad (2)$$

Arriving at this point the conception of the perturbation, which means a small deviation from the mean value of a variable is introduced. Further neglecting second order terms of perturbations restricts applications of the results of this derivation to situations in which the standard deviation of f = ln k is small, preferably smaller than 0.2, but at most 1 (ref. 5). Replacing each variable with the sum of its mean value, and the perturbation around this value, equation (2) becomes, with $f = \bar{f} + f'$ and $\phi = \bar{\phi} + \phi'$, after elimination of (2), expressed in mean values of the variables:

$$\frac{\delta^2\phi'}{\delta r^2} - \frac{c}{r}\frac{\delta f'}{\delta r} + \frac{1}{r}\frac{\delta\phi'}{\delta r} + \frac{1}{r^2}\frac{\delta^2\phi'}{\delta\theta^2} = 0 \qquad (3)$$

in which:

c = $Q/(2\pi\,\bar{k}_1\,D)$ ($-\frac{c}{r}$ is replacing $\frac{\delta\bar{\phi}}{\delta r}$) {L}

$\bar{k}_1$ = geometric mean of k {LT^{-1}}

D = depth of the aquifer {L}

Q = abstraction rate {L^3T^{-1}}

Also taken into account in (3) are : $\frac{\delta\bar{f}}{\delta r} = 0$, $\frac{\delta\bar{f}}{\delta\theta} = 0$, $\frac{\delta\bar{\phi}}{\delta\theta} = 0$. Equation (3) is called a stochastic differential equation as the variables have a stochastic character. Next, in order to derive the autocorrelation function of θ', (3) is expressed in independent variables r_1 and θ_1 (discerned from r_2 and θ_2 because their relative differences are used in the following):

$$\frac{\delta^2\phi'(r_1,\theta_1)}{\delta r_1^2} + \frac{1}{r_1}\frac{\delta\phi'(r_1,\theta_1)}{\delta r_1} + \frac{1}{r_1^2}\frac{\delta^2\phi'(r_1,\theta_1)}{\delta\theta'^2} = \frac{c}{r_1}\frac{\delta f'(r_1,\theta_1)}{\delta r_1} \qquad (4)$$

Multiplying left and right sides of the equation with f', at the coordinates r_2 and θ_2, equation (4) becomes after taking the expected value of each term: (ref. 6).

$$\frac{\delta^2 E\{\phi'(r_1,\theta_1)\ f'(r_2,\theta_2)\}}{\delta r_1^2} + \frac{1}{r_1}\frac{\delta E\{\phi'(r_1,\theta_1)\ f'(r_2,\theta_2)\}}{\delta r_1}$$

$$+ \frac{1}{r_1^2}\frac{\delta^2 E\{\phi'(r_1,\theta_1)\ f'(r_2,\theta_2)\}}{\delta\theta_1^2} = \frac{c}{r_1}\frac{\delta E\{f'(r_1,\theta_1).\ f'(r_2,\theta_2)\}}{\delta r_1} \qquad (5)$$

The expected values of $\phi'(r_1, \theta_1).f'(r_2, \theta_2)$ and of $f'(r_1, \theta_1).f'(r_2, \theta_2)$ are equal to the cross and autocorrelation functions of ϕ', f' and f', respectively, at the coordinates r_1 and θ_1, r_2 and θ_2. In short:
$E\{ \phi'(r_1, \theta_1)f'(r_2, \theta_2)\} = R_{\theta_1', f_2'}$, the cross correlation function of ϕ_2' at location 1 and f_2' at location 2;
$E\{ f'(r_1, \theta_1)f'(r_2, \theta_2)\} = R_{f_1', f_2'}$, the autocorrelation function of f_1' at location 1 and f_2' at location 2. Note $var(f'_1) = R_{f_1 f_1}$ and $covar\ (f'_1\ \phi'_1) = R_{f_1\ \phi_1}$
Equation (5) now becomes:

$$\frac{\delta^2 R_{\phi'_1 f'_2}}{\delta r_1^2} + \frac{1}{r_1}\frac{\delta R_{\phi'_1 f'_2}}{\delta r_1} + \frac{1}{r_1^2}\frac{\delta^2 R_{\phi'_1 f'_2}}{\delta \theta_1^2} = \frac{c}{r_1}\frac{\delta R_{f'_1 f'_2}}{\delta r_1} \qquad (6)$$

Using the same procedure the next equation can be derived also:

$$\frac{\delta^2 R_{\phi'_2\ \phi'_1}}{\delta r_2^2} + \frac{1}{r_2}\frac{\delta R_{\phi'_2\ \phi'_1}}{\delta r_2} + \frac{1}{r_2^2}\frac{\delta^2 R_{\phi'_2\ \phi'_1}}{\delta \theta_2^2} = \frac{c}{r_2}\frac{\delta R_{f'_2\ \theta'_1}}{\delta r_2} \qquad (7)$$

For solving the stochastic differential equation (3), $R_{\phi'_1 f'_2}$ and $R_{\phi'_1\ \phi'_2}$ must be solved out of the equations (6) and (7), into which equation (3) is elaborated. For this, knowledge of $R_{f_1',\ f'_2}$ is a fundamental requirement. The solution can be done in a numerical way, using the boundary condition $R_{\phi'_1 f'_2}$ as well as $R_{\phi'_1 \phi'_2} = 0$ for r is equal to R.

BASIC EQUATIONS FOR DERIVING THE STOCHASTIC PROPERTIES OF OTHER CHARACTERISTICS OF WELL FLOW.

It is evident other characteristics besides the drawdown are also influenced by the stochastic properties of the horizontal permeability. For briefness sake only the basic differential equations will be given here, that are derived in fundamentally the same way as equation (3), from which, with the help of numerical methods the stochastic properties of the variables can be derived. The differential equations are, with perturbations as variables:

$$b\frac{\delta q_r'}{\delta r} + \frac{1}{r^2}\frac{\delta^2 \phi'}{\delta \theta^2} = 0 \qquad (8)$$

$$r\frac{\delta q'_r}{\delta r} + q'_r + \frac{\delta q'_\theta}{\delta \theta} = 0 \qquad (9)$$

$$\frac{\delta^2 T'}{\delta r^2} - \frac{1}{r}\frac{\delta T'}{\delta r} - \frac{e}{2c}\frac{\delta^2 \phi'}{\delta \theta^2} = 0 \qquad (10)$$

in which:

b	$= 1/(\bar{k}_1 D)$	$\{TL^{-2}\}$
e	$= \mu . 2\pi/Q$	$\{TL^{-3}\}$
μ	= effective porosity	$\{-\}$
q'_r	= groundwater velocity in radial direction	$\{LT^{-1}\}$
q'_θ	= groundwater velocity in tangential direction	$\{LT^{-1}\}$
T'	= flow duration	$\{T\}$

Contrary to the above the expression from which the stochastic properties can be extracted for the deviations of a water particle from the mean radial flow path is more complex. This expression is only given with a reasonable approximation as:

$$\mathrm{var}(T') = \frac{\mu^2}{2(R-r_p)^2} \int_{r_p}^{R} \int_{r_p}^{R} \int_{0}^{2\Theta} \frac{(2\Theta - |\tau|)\, r_1 r_2}{R_{q'_\theta q'_\theta}(\tau, r_1, r_2)}\, d\tau\, dr_1\, dr_2$$

From the equation the relation between the angular displacement Θ and var(T') can be calculated for $r=r_p$, given $R_{q'_\theta q'_\theta}$. The value of var(T'), that can be derived from (10) can be used for solving Θ out of equation (11). This value of the angular displacement can be called the standard deviation of Θ, σ. The relation between the standard deviation of the angular displacement and the standard deviation of the tangential displacement s is given with a reasonable approximation by $\sigma^2(s) \approx r_p^2 . \sigma^2(\Theta)$, in which $s = r\,\mathrm{tg}\,\Theta$.

CONCLUSIONS

When using the perturbation method equations can be derived, which can serve as starting points for calculation of the stochastic properties of the characteristics of well flow, such as groundwater heads, flow velocities, flow duration and deviation of the flow of water particles from the mean radial flow path.

Further numerical elaboration will show the magnitude of the effect of the stochastical properties of horizontal permeability on the flow characteristics of well flow.

REFERENCES

1 K.J. Weber, Influence on fluid flow of common sedimentary structures in sand bodies, Society of Petroleum Engineers of AIME, SPE 9247, 1980.
2 D.W. Bennion and J.C. Griffiths, A stochastic model for predicting variations in reservoir rock properties, Trans AIME, 237 (1966) part 2, 9 - 16.
3 R.A. Freeze, A stochastic conceptual analysis of one-dimensional groundwater flow in non uniform homogeneous media. Water Resources Research, 11 (1975), 725-741.
4 P. Whittle, On stationary processes in the plane, Biometrika, 41(1954), 434-449.
5 L.W. Gelhar, Effects of hydraulic conductivity variations on groundwater flows. In: Hydraulic problems solved by stochastic methods, Proceedings Sec. Int. IAHR Symposium on Stochastic Hydraulics, Lund, Sweden, 1976.
6 A. Papoulis, Probability, Random Variables and Stochastic Processes. Mc. Graw-Hill Kogakusha, 1965.