7 INVENTORY MODELS

7.1 TYPES OF INVENTORY MODELS

Inventory theory deals with the problems of mathematical economic models that describe and solve problems connected with the storage of raw materials, intermediate goods, finished products, spare parts, etc. The aim of an inventory system is to balance the differences between production and demand that occur in almost all cases. Inventory theory concentrates on the problems of optimization of inventory systems. Investigation of unused redundant resources, e.g. storage, budget, labour, power production capacities, etc. (Walter and Lauber, 1975) is sometimes included in this theory.

Inventory theory does not apply a unique methodology, and it uses methods from various disciplines. The unifying element is the application of these methods to inventory problems and their optimization.

The basic feature of an inventory system is some space for storage or stock (usually of a finite capacity) that is required to meet the demand (supply of items – system output), and inventory replenishment (system input).

The construction and maintenance of stock requires funds – i.e. costs for setting up and holding the inventory. The objective of an optimal inventory policy is the minimization of overall costs.

The aim of the inventory model is an optimal inventory policy with various constraints. Various tasks may be investigated; e.g. the capacity of stock is given and the main goal is to determine the optimal system output (supply of items, dependent production or consumption process). Highly complex inventory models can occur if the items are stocked at several locations with various interrelationships; an inventory system is formed in this way. Difficulties occur if the stock can serve more than one purpose (this might apply to several kinds of items). The investigation of these systems requires great simplification to solve the problem analytically.

Inventory models are often classified according to the nature of the system variables. If no random effects occur in the inventory process, a single mathematical description is possible and the models are called *deterministic*. In stochastic models, on the other hand, random effects occur, and the models can be further classified by their stochastic characteristics. If they are time independent, the models are *stationary*; if time dependent, they are *non-stationary*, the characteristics of which are functions of time.

An important aspect of inventory models is the time factor. The model without an explicit time effect is called *static*. Usually it is an approximation of reality. A better solution of problems is given by a dynamic model which includes time-dependent relationships. Complex inventory models often require *dynamic* stochastic models.

Inventory models can be classified according to possibilities in the regulation of replenishment and supply, depending on the stochastic nature of one or both these features. All models can be classified according to the nature of parameters into discrete and continuous models.

Like models of operation research, inventory models can be classified by the methods used to find a solution. Most important are two groups of models: analytical and simulation models. The advantage of the latter group is the possibility of modelling complex situations, especially with random variables. Simulation modelling of inventory systems is performed by high-speed electronic computers using special simulation languages.

Different mathematical models can be used in inventory theory. Their choice depends on the result of systems analysis of the problem and its character.

7.2 DETERMINISTIC INVENTORY MODELS

A characteristic feature of deterministic models is a unique description of the inventory process where no random input occurs. Solution of a deterministic problem is often easy but differences occur, depending on the type of model, in relation to maximum storage, e.g. its time dependence and variability of the replenishment, interval, etc. (Dráb, 1973; Wagner 1973).

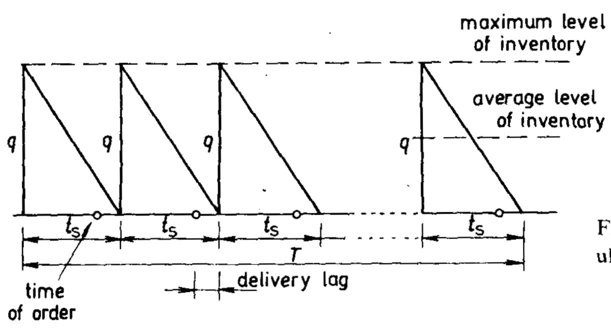

Fig. 7.1 An ideal time schedule of inventory

The simplest model, with an ideal replenishment and state of inventory, is given in Fig. 7.1. The maximum inventory is constant; demands of customers do not change and the replenishment frequency and delivery time-lag remain constant. In reality these characteristics can take different values.

Figure 7.2 illustrates examples of inventory situations when the supply from the inventory depends on the demand of customers and is approximated by different functions: an outline of a nonexhausted inventory is given in Fig. 7.2a. If this situation is repeated regularly in the period investigated, the capacity of stock is greater than necessary and not effective. Figure 7.2b shows that an inventory level can vary as a result of uneven replenishment or withdrawal. In Fig. 7.2c an important situation is described, i.e. when the inventory was exhausted before replenishment; in period t_2 losses occur (e.g. losses in production). This replenishment policy may be deliberate in a model with a non-linear relationship, if a small loss at the end of the replenishment cycle is compared with the greater cost of increasing stock capacity.

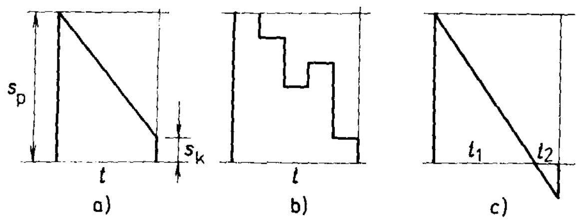

Fig. 7.2 Characteristic examples of inventory situations

This problem is solved by economic analysis which determines which losses or penalties due to exhausted inventory are acceptable as compared with the increased cost of a higher maximum inventory level.

The solution of problems connected with the inventory process is illustrated in an example of a simple deterministic model. In the plan in Fig. 7.1, it is assumed that the stock is regularly replenished by items in batches containing q items. During time T the amount Q is supplied in Q/q replenishments of items, scheduled at intervals $t_s = T/(Q/q)$.

The overall costs of the inventory process are composed of inventory holding costs and the cost of replenishment of the batch of q items (reorder costs). Inventory holding costs can be expressed in the following way:

The average inventory level is $q/2$, assuming a regular, continuous demand. The costs of storing one item are C_1 per unit of time. The inventory holding costs for time t_s are

$$\frac{q}{2} C_1 t_s \tag{7.1}$$

and during the whole production period T

$$\frac{q}{2} C_1 T \tag{7.2}$$

The costs for delivery of one batch of q items are C_s. The costs for delivery of Q/q batches are

$$C_s \frac{Q}{q} \tag{7.3}$$

The total costs are the sum of expressions (7.2) and (7.3)

$$N = \frac{q}{2} C_1 T + C_s \frac{Q}{q} \tag{7.4}$$

Figure 7.3 illustrates the relationship of functions (7.2), (7.3) and (7.4) to batch q.

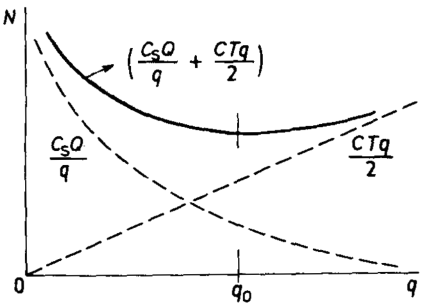

Fig. 7.3 Relation between the total costs and the batch size

The batch size q_0 that minimizes costs, is given by the zero value of the first derivative of function $N(q)$ with respect to q

$$\frac{dN}{dq} = \frac{1}{2} C_1 T - \frac{1}{q^2} C_s Q = 0 \tag{7.5}$$

It is solved for q_0

$$q_0 = \sqrt{\frac{2C_s Q}{C_1 T}} \tag{7.6}$$

Minimum costs N_0 are obtained by the substitution of (7.6) into (7.4)

$$N_0 = \frac{q_0}{2} C_1 T + \frac{C_s Q}{q_0} = \sqrt{2C_1 C_s Q T} \tag{7.7}$$

From the basic relationship for the replenishment interval t_s

$$t_s = \frac{T}{Q} q$$

the optimal interval t_0 that corresponds to minimum costs is obtained by substitution of q_0 for q

$$t_0 = \frac{T}{Q} q_0 = \sqrt{\frac{2C_s T}{C_1 Q}} \tag{7.8}$$

Equations (7.6) and (7.8) show that variables q_0 and t_0 depend on the ratio of costs C_s/C_1 (dimensionless value) and on ratio Q/T or its reciprocal value.

The type of model presented above explains the technique of model building and the method of derivation of optimal parameters that can be applied to more complex models (Ter-Manuelianc, 1968). These complex models can describe actual inventory systems in big organisations with automatic inventory control using modern electronic computers.

7.3 STOCHASTIC INVENTORY MODELS

Simple static or deterministic models are, as previously stated, a rough approximation of reality. Inventory problems of an obviously stochastic nature, e.g. the problems of the operational policy of WRS, cannot be treated by such simple models. Dynamic and stochastic models are necessary in such cases.

Dynamic inventory models with stochastic variation of inventory (Ter-Manuelianc, 1968; Hanssmann, 1962; Prabhu, 1965; Tersine, 1976) are characterized by resupplies (constant or variable) and by the probability distribution of future demand. Models of such problems include the following cost: build-up costs, inventory holding costs, penalty costs for an item out of stock, and inventory surplus costs.

In dynamic and stochastic models (as compared with static models), certain new features appear. For instance, the surplus inventory costs do not have only a negative impact as in static models, since the period of inventory holding is not assumed to be finite. Therefore, inventory surplus in one period can be compensated in the following period by a reduction in re-order. Therefore, the surplus inventory costs are equal to the increment of inventory holding costs.

The stochastic nature of demand results in variation in inventory consumption. If a stortage of inventory stock results in high penalty costs, the inventory level that corresponds to the average consumption is increased by safety (buffer) stock that should guarantee the inventory system against stochastic deviation of demand to higher values. The safety stock diminishes the risk of running out of stock, but it ties up the capital that otherwise could profitably be used, it requires inventory space, inventory maintenance, etc., and in brief, it represents additional inventory holding costs. It is clear that in dynamic stochastic models the penalty costs induced by a shortage of inventory stock should be balanced by the costs of safety stock.

In dynamic stochastic models the most important parameters are the intervals of replenishment. In a system with deterministic demand, the replenishment interval was computed without difficulty. On the other hand, in systems with stochastic demand the time required for replenishment is the main reason for the safety stock. The replenishment intervals can vary and during these intervals no items can be delivered. Therefore, increments in consumption during these intervals can be met only by safety stock.

The inventory policy in dynamic models consists of two types of decisions: the timing and magnitude of the resupply decision, i.e. indications when to replenish, and the amount to replenish.

In WRS, often a reversed alternative of this problem is considered. This alternative is also treated by inventory theory, and the solution of problems is analogous to that described below: usually the stochastic replenishment of items is assumed it corresponds to stochastic inflow into a reservoir), and the demand is governed by some rules (withdrawal of water from a reservoir according to some operational policy).

A further important difference between stochastic and deterministic models is the relationship between frequency and magnitude of orders that in deterministic models was given by the equation (7.9)

$$q = \frac{Q}{n} \tag{7.9}$$

where n is the frequency of orders Q/q
q – the batch size ordered,
Q – amount of demand during the period investigated.

In stochastic models where demand is a random variable, equation (7.9) holds true for mean values only. In individual periods there are random deviations in the actual demand from this mean value, and therefore a random deviation in the actual inventory level from the mean value occurs. The effect of demands on the actual inventory level must be balanced. This can be achieved by a change in the ordering frequency with constant batch size or by a change in batch size with a constant interval between replenishments. These procedures are the basis of the two main inventory policies in models with stochastic demand, viz., the Q-system and the P-system inventory policies, respectively (Ter-Manuelianc, 1968; Whithin, 1957; Wagner, 1962; Hadley-Within, 1963).

7.3.1 Q-System Inventory Policy

The Q-system (Fig. 7.4) uses a constant batch size, and variation in demand is balanced by changes in the frequency of orders. The safety stock is defined; it is used as a buffer for meeting demand during the delivery time-lag. As soon as the bulk of the stock has been used up, i.e. inventory reaches the signal level, a new batch is ordered and the safety stock is used till its arrival. Determination of the signal level takes into account the random variation of demand and penalties for items out of stock. As all batches are of a constant quantity, these systems are called Q-systems, where Q refers to the constant quantity. The safety stock is first refilled from the delivered batch and the remainder forms the bulk of the stock.

Variation in demand is met by the changes in the ordering frequency and therefore no safety stock is necessary for random increases of demand during the re-order cycle. It is sufficient to order the quantity that corresponds to average demand in the period investigated. If the actual demand is higher, the inventory level will drop sooner to the level of the safety stock and a new batch is ordered sooner. This automatic regulation of demand variation cannot be performed during the delivery time-lag. In Q-systems the safety stock safeguards the system from penalties due to items being cut of stock because of higher demand during the delivery time-lag.

The relationship between the level of the safety stock and the delivery time-lag is apparent from the equation for total inventory costs.

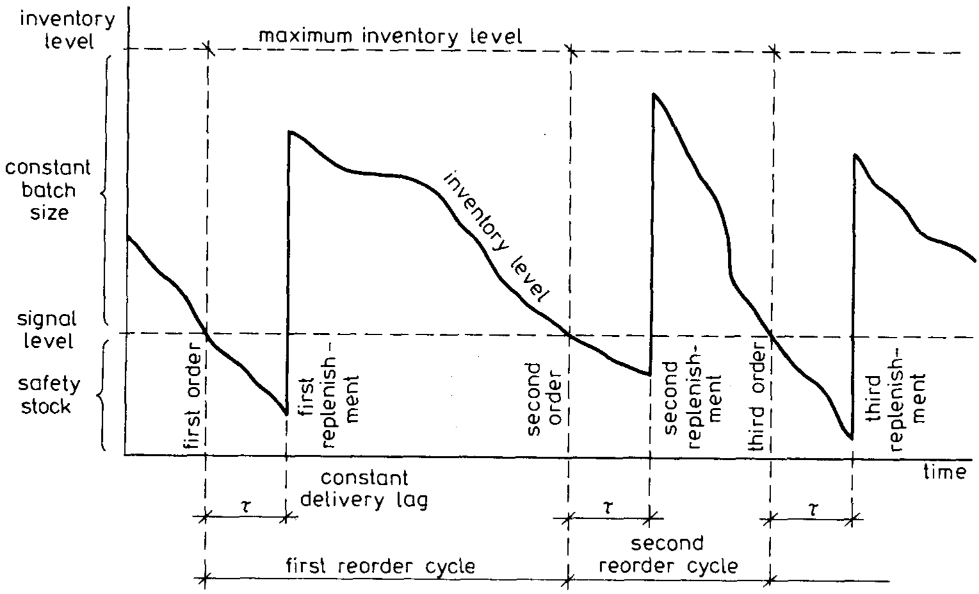

Fig. 7.4 Q-system inventory policy

The methods for the Q-system:

Notation

Q – the total demand of items per year,

q – the stochastic demand during the delivery time-lag with probability density $f(q)$,

z – safety stock in items,

C_s – costs of one batch,

c_1 – price of one unit in batch with costs C_s,

$c^{(1)}$ – annual inventory holding costs as a percentage of price of mean inventory level,

$c_1c^{(1)}$ – expected annual holding costs of one item,
C – penalty costs during the delivery time-lag (due to shortage of items); these costs do not depend on the level of item shortage,
t – mean duration of re-order cycle in years (expected interval between successive orders),
τ – delivery time-lag,
q_τ – average demand during delivery time-lag,
$1/t$ – number of re-order cycles per year,
C_s/t – costs of all batches per year.

Inventory holding costs for annual demand Q are

$$\frac{1}{2} Qc_1c^{(1)}t \tag{7.10}$$

Inventory holding costs of safety stock are

$$c_1c^{(1)}z \tag{7.11}$$

The computed costs of risk of inventory shortage for one re-order cycle are

$$C \int_{q_\tau + z}^{\infty} f(q)\, \mathrm{d}q \tag{7.12}$$

The costs of risk per year are

$$\frac{C}{t} \int_{q_\tau + z}^{\infty} f(q)\, \mathrm{d}q \tag{7.13}$$

The total annual inventory costs are given by the expression

$$N_{(t,z)} = \frac{C_s}{t} + \frac{1}{2} Qc_1c^{(1)}t + c_1c^{(1)}z + \frac{C}{t} \int_{q_\tau + z}^{\infty} f(q)\, \mathrm{d}q \tag{7.14}$$

The values of t and z that minimize the total costs were derived from equation (7.14) using the partial derivative with respect to z and taking its zero value; then t is expressed by:

$$t = \frac{C\, f(q_\tau + z)}{c_1c^{(1)}} \tag{7.15}$$

The value (7.15) is substituted into (7.14) and some rearrangement

$$[f(q_\tau + z)]^2 = \frac{2c_1c^{(1)}[C_s + C(1 - F(q_\tau + z))]}{C^2Q} \tag{7.16}$$

With the given probability density $f(q)$ the above equation can be used for computation of the optimum value of the safety stock $z = z^*$ with given values q_τ, c_1, $c^{(1)}$, C_s, C and Q, and from the equation (7.15) the optimum average length of the re-order cycle $t = t^*$ is determined. The optimum batch size is then determined by $o^* = Qt^*$.

7.3.2 *P*-System Inventory Policy

The second type of inventory policy in a dynamic and stochastic system is based on firm re-order time intervals with variable batch size. The batch size is determined on the following assumption: The sum of the batch size and the actual inventory level at the re-order point should equal the given quantity, i.e. the re-order inventory level, that is determined according to the variation in demand. The change in the batch size balances the variation of actual demand. The symbol *P* was derived from the word period as the model uses firm periods of re-order cycles (Fig. 7.5).

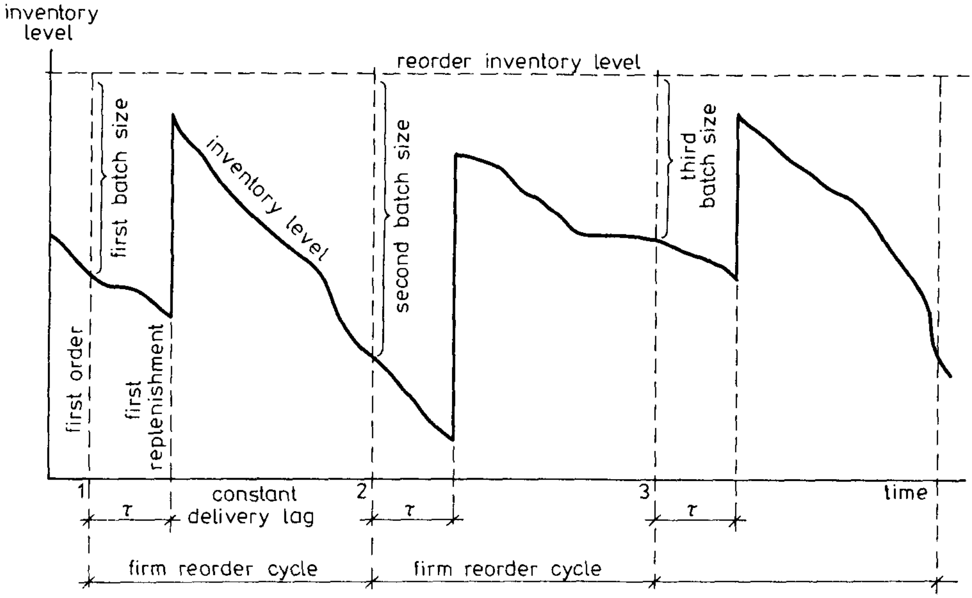

Fig. 7.5 *P*-system inventory policy I

In practice, the actual inventory level is monitored at regular intervals and the items are re-ordered accordingly. The model is used for the determination of the optimum re-order cycle duration and the optimum re-order inventory level, i.e. the level that should be reached at re-order point. The batch size is given as the difference between the re-order inventory level and the actual inventory level at re-order point. This quantity may be increased by the batch size of orders that have not yet arrived in stock. Such a situation (Fig. 7.6) is possible if the delivery time-lag is longer than the re-order cycle. The two main parameters, defining the *P*-system, are the duration of the re-order cycle and the re-order inventory level. The main difference between the *Q*-system and the *P*-system is in the method of dealing with a stochastic variation of demand. In *Q*-systems, a higher demand requires shortening of the re-order cycle, and the safety stock is used to meet the higher demand during the delivery time-lag. In P-systems, the safety stock has to cover variations in demand during the whole re-order period. There is some relationship between the batch size in one period and the risk of items being out of in all subsequent periods. For simplification of the P-system model, the safety stock is often determined in relation to one re-order cycle plus the following delivery lag.

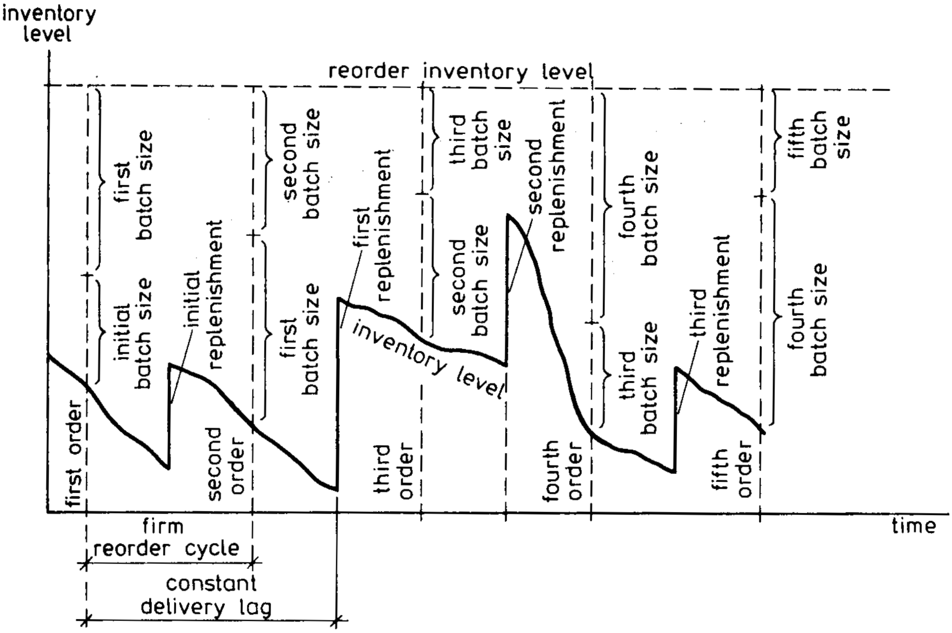

Fig. 7.6 P-system inventory policy II (if the delivery lag is longer than the reorder cycle)

The method of P-system optimization (notation is similar to that in the Q-system):

Q – the total demand of items per year,

q – demand per week (stochastic variable); one week was chosen as the basic time interval,

$f(q)$ – probability density of weekly demand q (more precisely v-th convolution of probability density),

C_s – costs of one batch,

c_1 – price of unit in batch with costs C_s,

$c^{(1)}$ – annual inventory holding costs as a percentage of price of the average inventory level,

$c_1c^{(1)}$ – annual holding costs of one item,

C – penalty costs during delivery time-lag (due to shortage of items); these costs do not depend on the level of item shortage,

t – duration of re-order cycle in weeks,
τ – delivery time-lag in weeks,
q_{week} – average demand per week,
z – safety stock.

The total annual inventory costs related to the duration of the re-order cycle and the safety stock are determined as follows:

$$N(t, z) = \frac{52C_s}{t} + \frac{1}{2} c_1 c^{(1)} t + c_1 c^{(1)} z + \frac{52C}{t} \int_{(t+\tau)q_{\text{week}}+z}^{\infty} f^{t+\tau}(q)\, dq \tag{7.17}$$

This equation can be used if the convolution of probability distribution is known. This condition can be fulfilled by some theoretical probability distributions only. Otherwise, some suboptimal P-systems are obtained because the basic parameters for given input values are determined by approximative calculations.

The method can be similar to approximations in Q-systems. The formula for the determination of optimum batch size, valid for a deterministic case, can be used for the determination of the optimum mean batch size, and this value is used for the duration of the re-order cycle. Then it is increased by the delivery time-lag, applying the convolution of probability distribution of demand to this period. The safety stock is determined by taking into account the inventory holding costs of safety stock, on the one hand, and the calculated costs of the risk of inventory shortage, on the other. The corresponding equation, including costs, for this case has the following form:

$$N(z) = c_1 c^{(1)} z + nC \int_{q_0+z}^{\infty} f^{t+\tau}(q)\, dq \tag{7.18}$$

where n – number orders per year,
q_0 – demand during the re-order cycle,
other symbols as previously defined.

The optimum level of the safety stock is computed by a similar method to equation (7.15), using the partial derivative with respect to z and taking its zero value:

$$f^{t+\tau}(q_0 + z) = \frac{c_1 c^{(1)}}{nC} \tag{7.19}$$

Up to this point, it has been assumed that the safety stock is always available. Actually, this inventory may be partly or completely exhausted. In that case, it is not correct to compute the inventory holding costs of the whole safety stock. If this fact is considered in the cost function, the expected average cost of the safety stock should be applied. In this way a further alternative of the P-system is formed. Similarly, other assumptions for the optimization of parameters of P-system inventory policies could be used.

7.3.3 Application of Markovian Stochastic Processes

The application of inventory theory is easier if a certain type of stochastic process is assumed, called the Markovian process – see Chapter 3 (Walter, 1970, 1973; Votruba and Nacházel, 1971).

A simple stochastic model of an inventory policy for a Markovian process will be demonstrated. The ordering points are fixed, the batch sizes differ; the P-system is therefore assumed. Two economically different inventory policies will be compared (see Table 7.1).

Table 7.1 Two inventory policies

Inventory level at the beginning of the week	Batch size	
	1st policy	2nd policy
0	1	2
1	1	0
2	0	0

The weekly demand is given by the following probability distribution:

demand q (in items)	0	1	2
probability $p(q)$ of occurrence of demand q	0.2	0.5	0.3

A weekly replenishment interval is assumed. The replenishment costs of one item are 30, of two items 60. Inventory holding costs of one item per week are 20. Sale of one item yields 100.

Evaluation of the first policy

The state of the system is defined by the inventory level at the beginning of the week. The re-order is placed at the beginning of the week, and it arrives at the beginning of the following week. First, the matrix of transition probabilities is set up, i.e. matrix

$$\mathbf{P} = \begin{Vmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{Vmatrix}$$

where e.g. p_{12} is the probability that the system will move from state 1 to state 2. For our case the matrix of transition probabilities is

$$\mathbf{P} = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0.8 & 0.2 \\ 0.3 & 0.5 & 0.2 \end{Vmatrix}$$

This matrix was constructed in the following way:

The system can move from state 0 to state 1 only if one single item is ordered in this state (see Table 7.1) and no demand can be met. State 1 gives more possibilities: $p_{10} = 0$ as the inventory may be exhausted, but one item was ordered, and it is not possible that the state will be 0 at the beginning of the next week. The system can move in two ways from state 1 to state 1 if demand is 1 or 2, then $p_{11} = 0.8$. The system can move from state 1 to state 2 on condition that demand is 0, then $p_{12} = 0.2$. Similarly, the last row was filled in.

As this transition matrix **P** is neither divisible nor periodic, a limiting state vector V (steady-state probability vector) exists. It is defined by the conditions:

$$V = V\mathbf{P}$$

and

$$\sum_{i=0}^{2} p_i = 1$$

where p_i are components of this vector (steady state or stationary probabilities).

Solution of the system

$$\begin{aligned} p_0 &= \qquad\qquad\qquad 0.3p_2 \\ p_1 &= p_0 + 0.8p_1 + 0.5p_2 \\ p_2 &= \qquad\quad 0.2p_1 + 0.2p_2 \\ p_0 &+ p_1 + p_2 = 1 \end{aligned}$$

produces the result

$$\begin{aligned} p_0 &= 0.06 \\ p_1 &= 0.75 \\ p_2 &= 0.19 \end{aligned}$$

The probabilities can be interpreted in the following way: During long-term activity of the system with the first inventory policy, the stock could be found to be zero at the beginning of the week in 6% of the weeks, and the inventory states 1 and 2 could be achieved in 75% and 19% of the weeks, respectively.

Next, the costs that result from the first inventory policy are computed. One item is ordered for states 0 or 1 that occur in 81% of the weeks. Then the expected weekly value of replenishment costs is $0.81 \cdot 30 = 24.3$.

The computation of the expected gross benefits for each possible state is as follows: if the system is in state 0, there are no inventory holding costs and no benefits from sold items; if the system is in state 1, then, with probability $p_{11} = 0.8$, one item will be sold that week, and with probability $p_{12} = 0.2$ one item will stay in stock all week. State 1 results in gross benefits

$$0.8 \cdot 100 - 0.2 \cdot 20 = 76$$

Similarly, for state 2 the gross benefits are

$$0.3 \cdot 200 + 0.5 \cdot 100 - 0.5 \cdot 20 - 0.2 \cdot 40 = 92$$

In each week the following gross benefits are expected for the individual states:

Initial state in the week	Expected benefits in the week
0	0
1	76
2	92

The computed steady-state vector determines how often these initial states occur in a long period. Therefore, the net weekly benefits for the first inventory policy are

$$0.06 \cdot 0 + 0.75 \cdot 76 + 0.19 \cdot 92 - 24.3 = 50.2$$

Evaluation of the second inventory policy (the method is similar)

The transition probability matrix is

$$\mathbf{P} = \begin{Vmatrix} 0 & 0 & 1 \\ 0.8 & 0.2 & 0 \\ 0.3 & 0.5 & 0.2 \end{Vmatrix}$$

The steady-state vector is

$$p_0 = 0.33$$
$$p_1 = 0.26$$
$$p_2 = 0.41$$

Using the second inventory policy, two items are ordered for the initial weekly state 0, which occurs with probability $p_0 = 0.33$. Replenishment costs of two items are 60. Therefore, the expected weekly replenishment costs are

$$0.33 \cdot 60 = 19.8$$

The expected gross weekly benefits are the same as with the first inventory policy. The expected net benefits of the second inventory policy are

$$0.33 \cdot 0 + 0.26 \cdot 76 + 0.41 \cdot 92 - 19{,}8 = 37.7$$

Since the first inventory policy yields higher weekly net benefits (50.2), it is more advantageous.

7.4 APPLICATION OF INVENTORY THEORY IN WRS

In principle, many problems of inventory control are similar to problems of WRS. The mathematical description of inventory issues of a stock with stochastic replenishment is analogous to that of active storage of a reservoir with stochastic inflow. Also, the inventory operational policy and operational rules for the release of water from reservoirs are similar (Fig. 7.7).

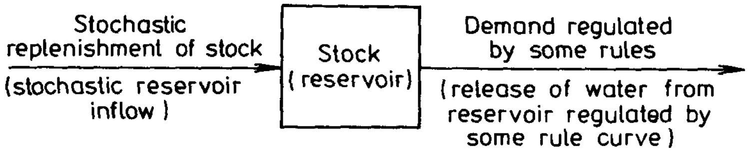

Fig. 7.7 Scheme of similarities of problems of the inventory theory and of WRS

The objective may be determination of an optimum capacity (e.g. storage of reservoir) with given demand (water requirement), investigation of optimum operating policy of inventory (reservoir) or determination of optimum reliability of an inventory system (WRS).

The mathematical similarity extends to other aspects of these problems, e.g. the inventory operating policy with stochastic demand for some items and water resource operational policy with stochastic demand for water.

These, and other possible analogies, are the basis of a series of approaches related to discrete or continuous time and items with finite or infinite inventory capacity. The common property of these problems is their random nature allowing application of the theory of probability.

Inventory theory and its applications have not been developed to such a degree as to solve the problems of complex WRS. It is confined to individual reservoirs or systems with a few reservoirs. The model presented in the following section is, therefore, a very rough approximation of a WRS, and the system is reduced to a system of a few reservoirs.

The mathematical background of inventory theory is similar to that of queue theory. Therefore, the application of both theories in WRS have similar characteristics and sometimes it is difficult to decide whether the WRS model is an application of inventory theory or queue theory. Some models will be described that use the principles of both theories, but they can scarcely be called applications of these theories. These models were developed by Kritsky, Menkel, Kartvelishvili, Cvetkov and others. In view of this fact, the model classification as inventory models or queue models is approximate but useful.

Methods of reservoir storage analysis based on inventory theory and queue theory and some other specialized methods can collectively be termed "the reservoir operational policy theory". The characteristic feature of this theory is its reflection of the stochastic nature of phenomena. It has produced many positive results, although it does not constitute a complete and finished theory. It is represented by a series of models, of various degrees of simplification, of WRS problems. The numerical solutions involved in the application of these methods are difficult, especially in models with many reservoirs, even if big computers are used. The Monte Carlo methods, i.e. numerical solution by repetitive computation using many realizations of stochastic processes for the solution of mathematical problems, can often help in such cases. Instead of an analytical solution, the Monte Carlo experiments are used for the determination of probability values.

The aim of the following examples of investigation of reservoir function by methods of the inventory, queue or similar theories is to determine the probability distribution of water storage at some time point, e.g. at the end of some chosen time interval (at the end of the year). The simplest models assume one reservoir, discrete time and discrete probability distribution; the more complicated models assume continuous time and probability distribution and a system of reservoirs.

Examples that greatly simplify the assumptions of WRS, have little significance for practical application. They are used as explanations of more complicated models. Complicated mathematical considerations leading to results that cannot be used in practice were omitted, and only methods of practical applicability have been concentrated on.

7.4.1 Operational Policy of a Single Reservoir at a Discrete Time

Firstly, the operational policy of a single reservoir is investigated (Moran, 1959). Inflow into the reservoir is a random variable. Release from the reservoir depends on the volume of water in the reservoir and the operational policy that determines the quantity and timing of the release. A reservoir with limited active storage is assumed. Time t is assumed in discrete time steps $t = 0, 1, 2 \ldots$ (Fig. 7.8).

Annual steps are assumed, and water is released once a year at a given fixed time. These simplifying assumptions are far from reality; they produce a rough approximation of the operation of a reservoir that is filled regularly in one season of the year (melting snow in the upper catchment area) and is released regularly in another season.

The total volume of water inflow in year t is denoted as X_t. Let us assume that X_t is a random variable at the interval $(0, \infty)$, it has a discrete probability distribution, and the values X_t for different t are mutually independent and have the same distribution[1]). The storage of water in the reservoir before the inflow X_t is denoted by Z_t. If $X_t + Z_t$ is greater than the active storage of the reservoir V, the volume $Z_t + X_t - V$ will be spilt uselessly. The distribution of this spill can be determined directly from the distribution of variables X_t and Z_t. At the end of the year water is released according to a fixed rule. The simplifying assumption is used that the same amount of water $M < V$, is released every year if this amount remained in the reservoir, or the amount $X_t + Z_t$, if $X_t + Z_t < M$. The storage Z_{t+1} remains in the reservoir.

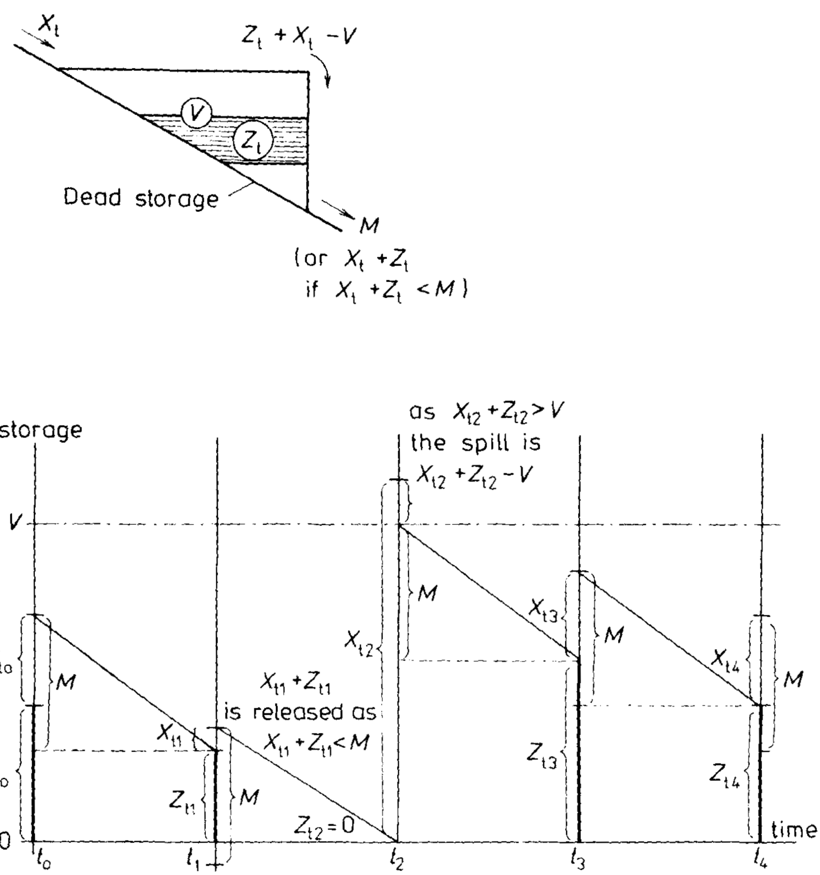

Fig. 7.8 Schematic representation of computation for the discrete time and the discrete probability distribution
V – reservoir active storage volume, X_t – reservoir inflow volume in the year t, Z_t – reservoir storage before the inflow X_t, M – release volume

[1]) The assumption of independence and constant distribution is not fulfilled in short time steps – e.g. in daily reservoir operation according to seasonal variability and serial correlation among daily flows.

The main task is the determination of the probability distribution of the variable Z_t (i.e. the storage in the reservoir at the end of interval t) for the stationary state of the stochastic process, defined by this system of rules, i.e. a state which is not dependent on the initial conditions. It can be proved that such a state of the process can be reached. This is an ordinary Markovian process, i.e. the conditional distribution of variable Z_{t+1}, given Z_t, does not depend on the past, i.e. the period before time t.

For numerical solution of this example a finite system of approximating linear equations can be used to determine the probability distribution of variable Z_t.

Let us assume that in time t X_t acquires the values 0, 1, 2, ..., with probabilities $p_0, p_1, \dots$. At the same time, variable Z_t acquires the values 0, 1, ... with probabilities $P_0, P_1, \dots, P_k$ and in time $t + 1$ it acquires the same values with probabilities $P'_0, P'_1, \dots P'_k$. It is also assumed that $V \geqq M$. Then the following system of equations is valid

$$
\begin{aligned}
P'_0 &= P_0(p_0 + p_1 + \dots + p_M) + P_1(p_0 + p_1 + \dots + p_{M-1}) + \dots + P_M p_0 \\
P'_1 &= P_0 p_{M+1} + P_1 p_M + \dots + P_{M+1} p_0 \qquad (7.20) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
P'_{V-M} &= P_0(p_V + \dots) + P_1(p_{V-1} + \dots) + \dots + P_{V-M}(p_M + \dots)
\end{aligned}
$$

Solution of this system gives the probability distribution of variable Z_{t+1}, given Z_t. We can then proceed to $Z_{t+2}, Z_{t+3}, \dots, Z_{t+n}$. These values form a Markov chain. For $n \to \infty$, a stationary solution is obtained if $p_i > 0$ $(i = 0, \dots, V)$.

Stationary probabilities are determined from (7.20) assuming $P'_i = P_i$ $(i = 0, \dots, \dots, V - M)$ and taking into account the relationship $\sum_{i=1}^{V-M} P_i = 1$. This task can be solved by matrix methods or (for big problems) by the Monte Carlo method. The computation of the simplified case of a single reservoir is presented. The same procedure can be used for models with several reservoirs and a more complex inflow pattern and operational policy. These more complicated cases require the use of computers.

The computation by the matrix method is performed for the following input values: active storage of the reservoir $V = 4$, release from reservoir $M = 2$. The discrete probability distribution of the reservoir inflow X_i $(X_i = 0, 1, \dots)$

$$
\begin{aligned}
p_0 &= 0.2 \\
p_1 &= 0.4 \\
p_2 &= 0.2 \\
p_3 &= 0.1 \\
p_4 + p_5 + p_6 + \dots &= 0.1 \\
\hline
\textstyle\sum p_i &= 1.0
\end{aligned}
$$

The aim of computation is the determination of the stationary probabilities P_i, $i = 0, 1, \dots, V - M = 2$ so that the storage at the end of the year will be $Z = 0, 1, 2$.

The input values are substituted in equations (7.20) and a set of $V - M + 1 = 3$ equations with three unknowns is obtained. The matrix of transition probabilities is

$$\mathbf{P} = \begin{Vmatrix} 0.8 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.4 \\ 0.1 & 0.2 & 0.4 \end{Vmatrix}$$

This matrix fulfils the conditions for the existence of a stationary state. To find the solution for the steady state, the following equations are used: $P_i' = P_i$ ($i = 0, 1, \ldots, \ldots, V - M$) and $\sum P_i = 1$. Then the following system will be solved

$$\begin{aligned} P_0 + P_1 + P_2 &= 1 \\ P_0 &= 0.8P_0 + 0.6P_1 + 0.2P_2 \\ P_1 &= 0.1P_0 + 0.2P_1 + 0.4P_2 \\ P_2 &= 0.1P_0 + 0.2P_1 + 0.4P_2 \end{aligned}$$

The resulting probabilities are $P_0 = 0.666$, $P_1 = 0.167$, $P_2 = 0.167$ so that storage at the end of the year will be $Z = 0, 1, 2$.

The problem can be solved, for example, by the Monte Carlo method in this way (Moran, 1959): Random two-digit numbers are used for the determination of transition probabilities from one state to another (i.e. for storage at the beginning and at the end of the year). Starting from zero, i.e. assuming an empty reservoir at the

Table 7.2 Monte Carlo method in tabular form

Hundreds of realizations of the random process	Observed frequencies of states		
	State (content of reservoir)		
	0	1	2
1	65	18	17
2	65	16	19
3	66	18	17
4	68	17	15
5	69	13	18
6	67	18	15
7	67	14	19
8	72	15	13
9	64	19	17
10	63	18	19
Sum of frequencies	666	166	168

beginning, the transition probabilities into states 0, 1, 2 are given by the first column of the given matrix. For example, if the first random number is A_1, the next state will be 0 if $A_1 \leqq 80$, it will be 1 if $80 \leqq A_1 \leqq 80 + 10$ or it will be 2 if $80 + 10 \leqq \leqq A_1 \leqq 80 + 10 + 10$.

In the next step, the column of the matrix of transition probabilities related to the state reached is used. The resulting values are presented in tabular form (Table 7.2). For example, ten sets with 100 realizations are used in every row and the resulting frequencies are listed. These frequencies approximate the required probabilities P_0, P_1, P_2. The sum of the frequencies in the last row gives a better approximation based on 1000 realizations. The number of realizations necessary for the required degree of accuracy is computed from the estimate of the standard error. In the simple case presented, a very good fit of accurate results and the frequencies for 1000 realizations are achieved.

7.4.2 Operational Policy in Continuous Time with Continuous Probability Distribution

The model in the previous section can be transformed by certain mathematical operations into a more general model that can better reflect reality by assuming continuous time and continuous probability distribution (Moran, 1959).

In this transformation the following procedure was used: First, a modification in the discrete model is carried out. We assume that in discrete probability distribution the sum of remaining storage in reservoir Z_t and the inflow X_t is Y_t, i.e. $Z_t + X_t = Y_t$. Let R_i denote the probability that $Y_t = i$. Moreover, we assume that the process is stationary, and that it does not depend on the initial conditions; with this assumption, the required probability distribution of Z_t can be found for a given distribution of $Y_t = Z_t + X_t$ and X_t. By substitution of R in (7.20) and rearrangement an infinite system of linear equations is obtained describing the distributions of random variables R_i:

$$R_0 = p_0R_0 + p_0R_1 + \dots + p_0R_M$$
$$R_1 = p_1R_0 + p_1R_1 + \dots + p_1R_M + p_0R_{M+1} \qquad (7.21)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$R_{V-M-1} = p_{V-M-1}R_0 + \dots + p_{V-M-1}R_M + p_{V-M-2}R_{M+1} + \dots + p_0R_{V-1}$$
$$R_{V-M} = p_{V-M}R_0 + \dots + p_{V-M}R_M + p_{V-M-1}R_{M+1} \dots + p_0(R_V + R_{V+1} + \dots)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$R_{V-M+S} = p_{V-M+S}R_0 + \dots + p_{V-M+S}R_M + p_{V-M+S-1}R_{M+1} + \dots$$
$$+ \dots + p_s(R_V + R_{V+1} + \dots)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

where, as in equations (7.20), M represents the amount of water released from the reservoir.

Since

$$P_0 = R_0 + \dots + R_M$$
$$P_{V-M} = R_V + R_{V+1} + \dots$$
$$R_{V-M+S} = 0 \quad \text{for } S > 0$$

the equations for the stationary state can be obtained from equations (7.21).

The case of continuous probability distributions is analogous. If the probability density function of a random variable is

$$X_t = f(x) \quad \text{for } x \geqq 0$$
$$X_t = 0 \qquad \text{for } x < 0$$

and the density function of $X_t + Z_t$ is $g(x)$, the following equation is valid:

$$g(x) = f(x) \int_0^M g(y)\, \mathrm{d}y + \int_M^V g(y)\, f(x + M - y)\, \mathrm{d}y + f(x - V + M) \int_V^\infty g(y)\, \mathrm{d}y \tag{7.22}$$

The three terms of the right-hand side of the equation correspond to the following conditions:

$0 \leqq X_t + Z_t \leqq M$ ... the reservoir is empty
$M < X_t + Z_t \leqq V$ ... the reservoir is not empty and no spillage occurs
$V < X_t + Z_t$ ... spillage is $(X_t + Z_t) - V$

The probability distribution of variable Z_t consists of three parts, a discrete component p_0 if $Z_t = 0$, the continuous probability density function $g\ (x + M)$ if $Z_t = x$ where $0 < x < V - M$ a discrete component p_1 if $Z_t = V - M$, and

$$p_0 = \int_0^M g(y)\, \mathrm{d}y \quad p_1 = \int_V^\infty g(y)\, \mathrm{d}y \tag{7.23}$$

Although the function $f(x)$ has a simple form, solution of equation (7.22) is complicated, and it has an analytical solution in strips of width M. Therefore, Moran and Prabhu (Moran, 1959) tried to find an explicit solution of equation (7.22) for a class of distributions that can describe the empirical distributions of flows. They concentrated their attention on Pearson's distribution type III and geometrical distribution. Although they used many sophisticated mathematical operations, they obtained solution for integer p only. Moran, therefore, suggested the application of the known results for Pearson's distribution type III with integer p with interpolation for other p. Kartvelishvili, 1963, argues that these types of distribution do no represent all actual cases.

Most difficulties in an analytical solution are caused by the boundary conditions $Z = 0$ and $Z = V - M$. Therefore, a possible simplification of computation is to assume a reservoir with infinite storage. Two cases are possible: a situation involving an almost full reservoir can be investigated if the probability distribution of variable Z gives very small probabilities that Z_t will approach zero. In this case a reservoir of infinite depth can be assumed. The opposite case assumes the very small probability of completely filling the reservoir, and the probability distribution of storage near the bottom of the reservoir is investigated with no limitation in the upper direction.

These problems, assuming an infinite reservoir storage, are beyond the scope of the application of inventory theory and are dealt with by queue theory (see section 8.6).

7.4.3 Operational Policy of a Single Reservoir with Carry-Over Storage in Discrete Time

The previous models applied inventory theory. Other methods (Kartvelishvili, 1967) are similar to these models; however, their classification as applications of inventory theory will be a simplification.

The origin of these models is older than inventory theory; they were developed independently of this theory, and as late as the last decade, some common features can be observed. These models are applied practically and compared with the previous simple models, they reflect more accurately the complexity of reservoir operational policy, taking into consideration its stochastic character.

The simplest model of a single reservoir can be described as follows: the annual flows and annual releases are assumed to be integer variables. This approximation is acceptable if the seasonal component is negligible in comparison with the carry-over storage of the reservoir. The losses are considered as a reduction in inflow or an increase in withdrawal. River flows are assumed to form a stochastic process with discrete time and an annual time step; the stochastic serial correlation of flows is neglected.

The model uses very simple assumptions, similar to those used in the first model. However, it has some advantages over Moran's model. It does not require the release to take place at a predetermined moment, and the release need not be the same every year. Relatively simple graphical methods of solution have been developed. Kartvelishvili emphasizes the methodological importance of this model, since the more general models of the theory of storage, with the release deterministic function of time or stochastic release function of time or in a form of operational rule (e.g. during reservoir filling), use this simple model of a single reservoir with carry-over storage as a starting point.

The method is explained graphically. Let α denote the annual release from the reservoir, and β the reservoir storage.

Both these values are related to the mathematical expectation of the annual flow. Further, $X(z) = 1 - F(z)$ is the reliability of the annual flow (in volume units), i.e. the probability that the ratio of its actual and expected values will not be less than z.

Let us start with a constant release α, i.e. α = const > β. Assume (Fig. 7.9) $OD = 1$, the $ABCD$ is the curve of reliability of filling up the reservoir at the end of the k-th year of operation. The ordinates of this curve are $\Theta_k(y)$.

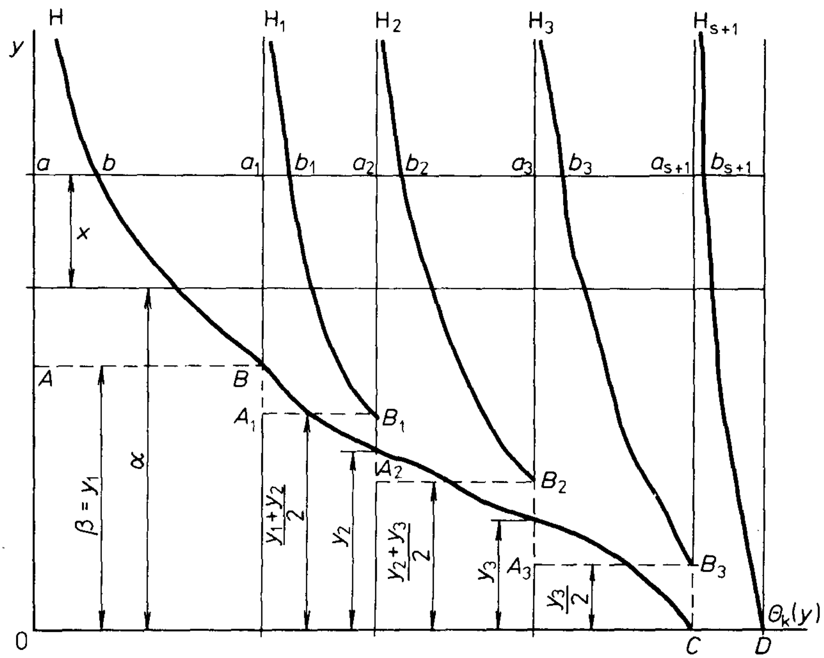

Fig. 7.9 The graphic method for the determination of the operation policy of a single reservoir with carry-over and discrete time

No serial stochastic relationship between flows is assumed; the filling up of the reservoir at the end of the year and the release in the following year are, therefore, stochastically independent. Using these assumptions, Kritsky and Menkel designed a method for the determination of function Θ_{k+1} of reliability of filling up of the reservoir at the end of the year $k + 1$. The curve BC is replaced by a graph with a finite number of steps, $BA_1B_1A_2 \ldots A_sB_sC$. Curves HB, $H_1B_1, \ldots H_sB_s$, H_{s+1} D are also constructed. Their functional values relative to points A, $A_1, \ldots, A_sC$ are equal to z and their abscissas relative to the same point are $AB_X(z)$, $A_1B_{1X}(z) \ldots A_sB_{sX}(z)$, $CD_X(z)$. Then with annual release α = const, the reliability of the reservoir filling up at the end of the year $k + 1$ by volume x $(0 \leqq x \leqq \beta)$ will be approximated by

$$\Theta_{k+1}(x) = ab + a_1b_1 + \ldots + a_sb_s + a_{s+1}b_{s+1} \tag{7.24}$$

It is apparent that abscissa ab is the probability that the reservoir content at the end of the year $(k + 1)$ will exceed the value x on condition that it was full at the end of the year k (the probability of the reservoir being full at the end of the year k is AB). The abscissa a_1b_1 is the probability that at the end of the year k the reservoir content was $(y_1 + y_2)/2$ etc. The abscissa a_{s+1} is the probability that at the end of the year $k + 1$ the reservoir content will exceed x on condition that it was empty at the end of the year k (the probability of the reservoir being empty at the end of the year is CD). Since the phenomena whose probabilities are given by abscissas ab, $a_1b_1, \ldots,$ $\ldots, a_{s+1}b_{s+1}$ are mutually exclusive, probability $\Theta_{k+1}(x)$ will be given by the sum of the corresponding probabilities, as expressed in equation (7.24). A precise analytical solution can be obtained by division of the curve BC into an infinite number of steps (Kartvelishvili, 1958). The reliability of a full reservoir was determined by a simple integral Fredholm equation, type II, the theory and solution of which are known.

This model of operational policy of a single reservoir was generalized by Kartvelishvili, 1967, for two or more reservoirs. It takes into account the stochastic relationship between reservoir inflows, but it does not assume a serial correlation in a one-reservoir site. This relatively simple model requires a system of complicated integral equations of probability distribution of reservoir filling.

Similar models of the same and related problems were investigated by Kritsky and Menkel, 1952, 1959; Kartvelishvili, 1967 and Reznikovsky, 1964. Some models are oriented towards hydropower; the objective of some models is the maximization of reliability, and, therefore, these models are, in fact, optimization models. One group of models includes seasonal operational policy.

All these models of operational policies with given release requirements (e.g. expressed by a probability function) involve the solution of integral equations or a set of these equations. In the simplest case (e.g. at the beginning of section 7.4.3) the solution of mathematical integral equations is the required probability characteristic of the operational policy. In other models the solution of these equations is the basis for the determination of these characteristics.

In principle, three different procedures are possible in practical computation:

– the direct solution of integral equations or a set of equations,

– the approximation of integral equations by a system of linear algebraic equations (e.g. by the methods of moments or by numerical integration),

– the Monte Carlo method.

All these procedures were described by Kartvelishvili, 1967. The application of the Monte Carlo method to these models was described by Svanidze, 1964, Reznikovsky, 1964, and Vicens, 1963.