10 OTHER METHODS OF WATER RESOURCE SYSTEMS ANALYSIS AND SYNTHESIS

10.1 THE COMBINATION OF THE OUT-OF-KILTER METHOD AND THE SIMULATION MODEL

A special method was devised in "Planning the Comprehensive Development of the Vistula River System", which was elaborated in the co-operation between Hydroproject Warsaw and the United Nations Development Programme (UNDP). The main objective of the program was to design an optimal investment program for WRS in the basin of the Vistula River up to the year 2000. The Project was multipurpose, and it included a number of models and programmes. From the methodological point of view, the combination of the simulation model and an allocation optimization model, using the Ford-Fulkerson "*out-of-kilter*" method, was the most interesting.

The WRS included water reservoirs, rivers and diversion channels, which served to transport water to the users. The model of this system was set up in the form of a network (graph) with flows in arcs of the network which corresponded to discharges or flows of water towards certain goals. These were, for example, the supply of water for various demands: minimum flow augmentation, or the unused part of flow. Each arc was assigned a coefficient related to the importance of water use. This coefficient was related to the potential losses that might occur due to a reduction in the water supply for this goal, and it was proportional to the unit loss due to the deficit of water for this goal (e.g., municipal water supply).

The allocation model is combined with the simulation model in the following manner. At each time step of the simulation model a vector of optimal flows is determined by the allocation optimization model. The water resources are allocated to the users in such a way as to minimize the weighted sum of deficits. The unit loss coefficients were used as weights.

10.1.1 The Out-of-Kilter Method

The principles of a special technique – an out-of-kilter optimization algorithm can be described briefly. The method starts by *labelling the nodes* with accompanied correction of flows in arcs of graph (network) to reduce the value of the objective function. The method is iterative and every feasible flow can be used as the starting state. Each arc of the graph has a limited capacity.

For clarifying the method and its assumptions let us consider the following linear problem in this form:

$$a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n = b_i \quad \text{for } i = 1, 2, \ldots, m \tag{10.1}$$
$$d_j \leqq x_j \leqq h_j \quad \text{for } j = 1, 2, \ldots, n \tag{10.2}$$
$$c_1x_1 + c_2x_2 + \ldots c_nx_n = \min \tag{10.3}$$

where $a_{i,j}$, b_i, d_j, h_j, c_j are components of a vector of constants, the equation (10.1) describes the form of the graph, the equation (10.2) gives the lower and upper bounds of the arc flow and the equation (10.3) is the objective function. Assume that $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is the vector of variables that satisfy the equations (10.1) and (10.2), i.e., vector $\boldsymbol{x}$ is a feasible solution and assume that $\boldsymbol{z} = (z_1, z_2, \ldots, z_m)$ is the vector of dual prices that meet the implications

$$c_j + (z_1a_{1,j} + \ldots + z_ma_{m,j}) > 0 \rightarrow x_j = d_j \tag{10.4}$$
$$c_j + (z_1a_{1,j} + \ldots + z_ma_{m,j}) < 0 \rightarrow x_j = h_j \tag{10.5}$$

for all j. Using the dual theorem of linear programming and further relationships it can be shown[1]) that vector $\boldsymbol{x}$ is the minimizing solution and that equations (10.4) and (10.5) can be used as the optimality criteria.

If vector $\boldsymbol{x}$ satisfies condition (10.1) and if:

$$s_j = z_1a_{1,j} + z_2a_{2,j} + \ldots + z_ma_{m,j} \tag{10.6}$$

then for j-th component of vector $\boldsymbol{x}$, i.e. x_j, the following combinations of relationships among c_j and s_j, on one hand, and d_j, x_j, h_j, on the other, are possible. These combinations classify x_j in an exhaustive manner:

$$c_j + s_j > 0; \; x_j = d_j \tag{10.7}$$
$$c_j + s_j = 0; \; d_j \leqq x_j \leqq h_j \tag{10.8}$$
$$c_j + s_j < 0; \; x_j = h_j \tag{10.9}$$
$$c_j + s_j > 0; \; x_j < d_j \tag{10.10}$$
$$c_j + s_j = 0; \; x_j < d_j \tag{10.11}$$
$$c_j + s_j < 0; \; x_j < h_j \tag{10.12}$$
$$c_j + s_j > 0; \; x_j > d_j \tag{10.13}$$
$$c_j + s_j = 0; \; x_j > h_j \tag{10.14}$$
$$c_j + s_j < 0; \; x_j > h_j \tag{10.15}$$

[1]) See Ford and Fulkerson, 1962; Yermolyev and Melnik, 1968; Busacker and Saaty, 1968; Harrary, 1969.

If all components of vector $\boldsymbol{x}$ satisfy equations (10.7), (10.8) or (10.9), then the solution if *feasible and optimal.* If any flow in arc x_j satisfies any one of the three conditions (10.7), (10.8) or (10.9), we say that the arc is in-kilter; otherwise, the arc is out-of-kilter.

The algorithm looks for some component that is out-of-kilter and step by step it is transformed to the in-kilter state, all the remaining components staying in-kilter. Other components that are out-of-kilter do not change, or they may be improved. The algorithm is used and converges for integer numbers. All the input values are given as integers.

The claim for a feasible starting solution of the iterative method does not complicate the solution as a *trivial solution* with all components of $\boldsymbol{x}$ equal to zero will do (with $d_j = 0$ for all j). If a feasible non-zero solution is known, than the time for a solution to be reached is greatly reduced. Therefore, this method is advantageous in a set of tasks where only certain parameters change.

The graph is described by the *set of arcs and nodes.* Each arc is directed and determined by starting node i and terminal node j[1]).

Three numbers are assigned for every arc (i, j): $d_{i,j}$ (lower bound of flow), $h_{i,j}$ (upper bound of flow or capacity) and $c_{i,j}$ (costs or cost values); the following constraints must be satisfied

$$0 \leqq d_{i,j} \leqq h_{i,j}; \qquad 0 \leqq c_{i,j} \tag{10.16}$$

For the description of the graph and solution of optimal flow the term *circulation* is introduced, which simplifies the model. Circulation is a non-negative integer vector $\boldsymbol{x} = (x_{i,j})$ having for each arc (i, j) one component and meeting the condition of mass balance for each node i, i.e.

$$\sum_{j=1}^{m} (x_{i,j} - x_{j,i}) = 0 \quad \text{for} \quad i = 1,2, \dots, n \tag{10.17}$$

If this circulation satisfies the condition

$$d_{i,j} \leqq x_{i,j} \leqq h_{i,j} \quad \text{for all arcs } (i, j) \tag{10.18}$$

than $\boldsymbol{x}$ is feasible circulation. The component of circulation $x_{i,j}$ is the flow in arc (i, j).

A feasible circulation that gives a minimum sum is

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j} x_{i,j} \to \min \tag{10.19}$$

[1]) In equations (10.1 – 10.15) index j denotes the number of arc. In the following text i and j denote the number of nodes, therefore, the data for arcs must be assigned by two indices i and j. This change of notation is necessary for the introduction of the notion of circulation.

and it is the *optimal solution* with minimum losses. Therefore the optimum circulation is sought. First of all it is necessary to find out if there is a non-trivial solution. This non-trivial solution exists if, and only if, the inequalities

$$\sum_{\Omega} h_{i,j} \geqq \sum_{\Omega} d_{i,j} \tag{10.20}$$

are satisfied. The sum in inequality (10.20) concerns set Ω, i.e., the set of arcs (i, j). where i is the element of set L and j is the element of its supplement $\bar{L}$; L is the subset of the set of all nodes. Condition (10.20) is to be satisfied for all subsets L.

Using the circulation model, the optimal conditions are equivalent to conditions (for all arcs (i, j))

$$c_{i,j} + z_i - z_j > 0 \rightarrow x_{i,j} = d_{i,j} \tag{10.21}$$

$$c_{i,j} + z_i - z_j < 0 \rightarrow x_{i,j} = h_{i,j} \tag{10.22}$$

where $z = (z_i)$ is the vector of integer numbers with one component for each node i, called node costs (Ford and Fulkerson, 1962).

If $\boldsymbol{x}$ is a feasible circulation and if z exists and satisfies implications (10.21) and (10.22), then the circulation $\boldsymbol{x}$ is optimal.

Denoting

$$f_{i,j} = c_{i,j} + z_i - z_j \tag{10.23}$$

and introducing the *out-of-kilter number*, which equals zero for in-kilter states and is non-negative for out-of-kilter states, the following list of states and out-of-kilter numbers (state numbers) is obtained:

State		State number	
(a) $f_{i,j} > 0$	$x_{i,j} = d_{i,j}$	0	
(b) $f_{i,j} = 0$	$d_{i,j} \leqq x_{i,j} \leqq h_{i,j}$	0	
(c) $f_{i,j} < 0$	$x_{i,j} = h_{i,j}$	0	
(a_1) $f_{i,j} > 0$	$x_{i,j} < d_{i,j}$	$d_{i,j} - x_{i,j}$	
(b_1) $f_{i,j} = 0$	$x_{i,j} < d_{i,j}$	$d_{i,j} - x_{i,j}$	(10.24)
(c_1) $f_{i,j} < 0$	$x_{i,j} < h_{i,j}$	$f_{i,j}(x_{i,j} - h_{i,j})$	
(a_2) $f_{i,j} > 0$	$x_{i,j} > d_{i,j}$	$f_{i,j}(x_{i,j} - d_{i,j})$	
(b_2) $f_{i,j} = 0$	$x_{i,j} > h_{i,j}$	$x_{i,j} - h_{i,j}$	
(c_2) $f_{i,j} < 0$	$x_{i,j} > h_{i,j}$	$x_{i,j} - h_{i,j}$	

The arc (i, j) is in-kilter if it is in one of the first three states, i.e., (a), (b), or (c). For arcs out-of-kilter, the out-of-kilter number for states (a_1), (b_1), (b_2), (c_2) gives the degree of deviation of the arc flow $x_{i,j}$ from the optimum, the out-of-kilter number for states (a_2) and (c_1) measures the deviation from conditions (10.21) and (10.22). The out-of-kilter algorithm ensures that all the out-of-kilter numbers do not increase, i.e., they remain the same or decrease. This was developed by Ford and Fulkerson, 1962 and a program in FORTRAN is included in CMEA program 1A.301. The description of this algorithm and an example of its numerical application is given in section 10.4.

10.1.2 Application of Out-of-Kilter Algorithm

The algorithm described is used for optimal allocation of the amount of water that is available in a time period t. If there are no reservoirs in the system, the decisions on optimal allocation in particular periods are mutually independent, and the total optimum is the sum of optimal solutions for individual periods. If some reservoirs are included in the system, some amount of water inflow inperiod t can be maintained for future periods. Then the total optimum is no longer the sum of optimal allocations in individual periods. According to the fact that users with different costs values (unit losses due to deficits) are in the system, it may be reasonable to admit small deficits in period t for users with low cost values in order to reduce the losses in periods $t + n$ for users with high cost values. In simulation models in the General Water Plan of Czechoslovakia this problem was solved heuristically preventing the losses due to deficits for users with high cost values in a deterministic simulation model.

In WRS in the basin of the Vistula River at first a solution on the assumption of known future flows was obtained. The *length of period* that should be taken into account in the optimization procedure was investigated. It is apparent that this length will be related to the operation cycles of reservoirs, and that it can include 12 months or more. The practical experience gained in the investigation of WRS in the basin of the Vistula River have shown that six months are sufficient.

The allocation problem in this model was solved not only for the current month, but also for the following months, six in total. Then changes in the system (the changes of storage of reservoirs included) were carried out using the simulation model, and the allocation problem was solved for the subsequent period.

10.1.3 Operating Rules Determination by a Stochastic Simulation Model

In simulation runs with six months in out-of-kilter procedure the value of the objective function (giving the sum of losses) was significantly lower than in simulation runs with only one period in out-of-kilter procedure. The former value of objective function is probably very far from the value for actual operation when the estimate of future flows is uncertain. Therefore, other methods were tested that were more realistic in system operation. First, it was assumed that the value of flow was known for the current period, and for future values *modal values* were used. If an observation for a longer period was available, a more reliable forecast could be formulated based on regression to past flows[1]).

For the computation of regression relationships, Kindler (1975) developed a method using the *stochastic simulation model*. As input it uses a multi-dimensional model, derived by Matalas for a first-order Markov chain and log-normal transformation. In the stochastic simulation model, the allocation is performed by the out-of-kilter method in each period, using the known flows for the future months, i.e., using a total of six months. For each month the storage at the end of the monthly period is related to the storage at the beginning of this period and to inflows in the system. Dropping the index m indicating that all values in equation (10.25) relate to month m, the regression equation has the following form:

$$V_j' = b_{0,j} + b_{1,j}Q_1 + b_{2,j}Q_2 + \dots + b_{n,j}Q_n + b_{n+1,j}V_1 + \dots + b_{2n,j}V_n \qquad (10.25)$$

where $b_{i,j}$ are regression coefficients, Q_i is the inflow in system for reservoirs 1 to n, V_i is the initial storage of reservoirs 1 to n (i.e., at the beginning of month m), and V_j' is the final storage of reservoir j (i.e., at the end of month m; n equation for each reservoir $j = 1$ to n).

In the actual operation of reservoirs the inflows for the current period are presumed to be known, however, the operation rule in WRS is defined by equation (10.25) for each reservoir $j = 1, 2, \dots, n$ and for each month of the year m. The resulting value of the objective function is greater than in the case of an ideal forecast, however, it is better (by approx. 30% over the example given by Kindler, 1975) than in calculations without any forecast.

10.1.4 Out-of-Kilter Method Parameters

The basic assumption of the successful application of the out-of-kilter method in practice is the appropriate estimation of *cost values in arcs* of the graph. It means proper estimation of the losses due to deficits for different users. This problem was partly solved in the Vistula River Project using the following approximations:

– the municipal water supply was taken as highest priority withdrawal and the cost values were taken as prohibitive to prevent deficits for this goal,

– the industrial water supply was evaluated by the relationship between the product and the amount of water supplied,

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[1]) In the WRS Project in the basin of the Vistula River, the observations of flows were only available for the period 1951 – 1965, which was not sufficient for the determination of regression coefficients in forecast.

– the coefficient for water supply for irrigation was determined by the relationship of annual production increased by irrigation with different values for the monthly periods,

– the coefficients for minimum flow augmentation were determined by lowest values in accordance with the overall tendency to minimize the investment costs,

– the coefficients for river flow regulation for water power plants were determined by the value of power production.

It is apparent that these coefficients were used as an ordering principle for priority determination. They are not ideal as they cannot reflect the sensitivity of users in industry to water deficits, the depth of eventual deficits, their relative losses, etc. Nevertheless, the allocation optimization using these approximations meant significant progress for the solution of WRS problems and the combination of simulation models with the out-of-kilter algorithm was successful for the optimization of WRS.

The method was applied to a practical task. However, further work is necessary and some problems need to be solved, mainly in the determination of cost values and in the method of flow forecast for WRS operation policy (Kos, 1979b).

10.1.5 Examples of a Numerical Solution by the Out-of-Kilter Method

As a simple example, illustrating the out-of-kilter method, the flow optimization in the network in Fig. 10.1., is shown. The input and output values of flows and the optimal flows in arcs are given in Table 10.1. For clarification of the term circulation, an example of the circulation in node 6 is given in Fig. 10.2 for the network in Fig. 10.1.

An example of a practical application of this method in WRS is presented by Kindler (1975). This WRS includes three reservoirs with approximative active

Table 10.1 The optimal flows in arcs of the simple network

Number of arc j	Cost value c_j	Optimal flow x_j	Number of node i	Inflow (+) or outflow (–)
1	10	25	1	– 5
2	11	10	2	–15
3	15	15	3	–45
4	20	20	4	+15
5	40	0	5	+20
6	30	30	6	+30
7	40	0		
8	49	0		

storages of 0.346 km^3, 0.225 km^3 and 0.297 km^3. The out-of-kilter method requires that all the data should be expressed in the same units, i.e., the flow in arcs (in this case tens of litres per second in an average month with 2,629,800 seconds). Therefore the reservoir storages were transformed to these flow units, and they are 13 156 (i.e., 132 $m^3 s^{-1}$ per month), 8556 and 11 294, respectively. The initial storage of reservoirs is assumed as the half of active storage.

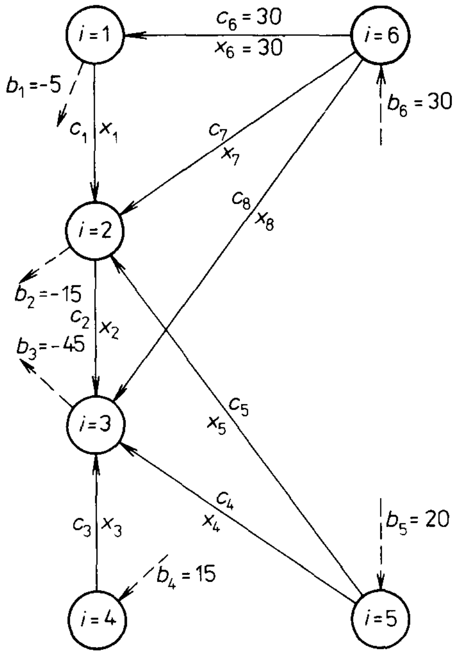

Fig. 10.1 Graph of a simple network
$i = 6$ – number of node, c_j – cost value (for one unit of flow x_j), d_j, h_j – lower and upper bounds of flow in arc j, x_j – flow in arc j
Balance in node 6: $x_6 + x_7 + x_8 = 30$; i.e. $a_{6,j} = 0$ for $1 \leqq j \leqq 5$, $a_{6,j} = 1$ for $6 \leqq j \leqq 8$, $b_6 = 30$

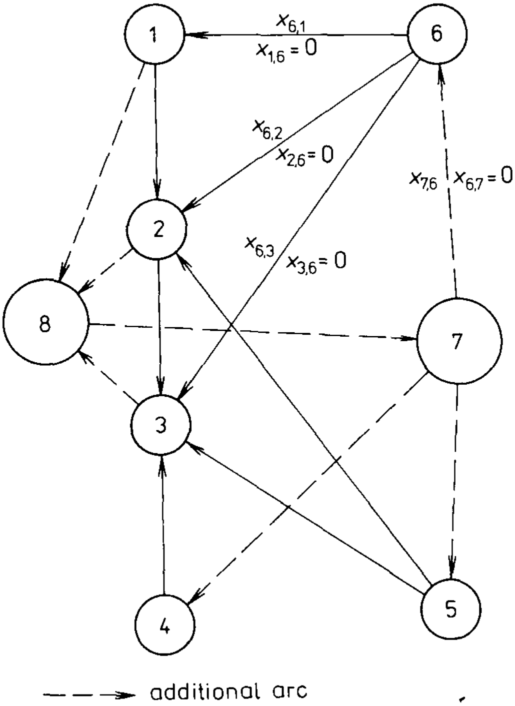

Fig. 10.2 Circulation in node 6 of the simple network
Equation (10.17) for node 6:
$x_{6,1} + x_{6,2} + x_{6,3} + x_{7,6} = 0$
node 7 is the source, node 8 is the sink

The goals of the WRS are the municipal and industrial water supply in five industrial centres ($a-e$), minimum flow augmentation at some points of the system. A schematic representation of WRS is given in Fig. 10.3. The transformed representation in the form of a graph is given in Fig. 10.4. The numbers of nodes in both schematic representations are equal to help a comparison between the two figures.

Table 10.2 Matrix description of the graph

No. of arc	From node to node		Bounds of flow lower	upper	Unit loss	Remark
1	2	3	4	5	6	7
1	*2*	*6*	*0*	999 999	0	River flow
2	*2*	*1*	0	1000	0	Transbasin diversion
3	*1*	*4*	0	999999	0	River flow
4	*3*	*14*	0	999999	0	River flow
5	*0*	*10*	0	0	0	Tributary flow
6	*0*	*15*	0	0	0	Tributary flow
7	*0*	*17*	0	0	0	Tributary flow
8	*1*	*0*	120	120	0	Reservoir losses
9	*4*	*5*	0	1000	0	Inflow
10	*4*	*12*	0	50	−10	Minimum flow
11	*4*	*12*	0	999999	0	River flow
12	*0*	*12*	58	58	5	Waste flow
13	*2*	*0*	41	41	0	Reservoir losses
14	*6*	*7*	0	900	0	Inflow
15	*7*	*5*	0	550	0	Inflow
16	*7*	*9*	0	250	−1000	Water requirements
17	*4*	*9*	0	90	−1500	Water requirements
18	*9*	*12*	0	350	0	Waste inflow
19	*7*	*8*	0	75	−1500	Water requirements
20	*8*	*12*	0	390	0	Waste flow
21	*6*	*10*	0	230	−10	Minimum flow
22	*6*	*10*	0	999999	0	River flow
23	*0*	*10*	16	16	0	Waste flow
24	*10*	*11*	0	590	0	Inflow
25	*11*	*8*	0	590	−1000	Water requirements
26	*10*	*12*	0	300	−10	Minimum flow
27	*10*	*12*	0	999999	0	River flow
28	*5*	*13*	0	1390	−5000	Water requirements
29	*13*	*12*	0	910	0	Waste flow
30	*12*	*16*	0	999999	0	River flow
31	*3*	*0*	59	59	0	Reservoir losses
32	*14*	*15*	0	190	−10	Minimum flow
33	*14*	*15*	0	999999	0	River flow
34	*0*	*15*	25	25	0	Waste flow
35	*15*	*11*	0	600	0	Inflow
36	*15*	*16*	0	210	−10	Minimum flow
37	*15*	*16*	0	999999	0	River flow
38	*15*	*16*	0	100	−1500	Water requirements
39	*16*	*17*	0	999999	0	River flow
40	*15*	*0*	0	600	−500	Water requirements
41	*17*	*0*	0	999999	0	River flow
42	*17*	*0*	0	1900	−10	Minimum flow

The river flow is reduced by the value of minimum flow if the node is connected by an additional arc for minimum flow. For arcs *5*, *6*, *7* the lower and upper bounds are given for each period.

The definition of the graph with a description of the flows in the arcs is given in Table 10.2. The arcs are defined by the starting and terminal nodes, lower and upper boundaries and cost value, which equals the unit loss. In each node of the graph (network) the circulation constraints must be satisfied, i.e., mass balance in the node. In nodes where water is consumed (the mass balance would not be met if the surface flow alone is assumed) the evaporation, infiltration or transbasin transfer

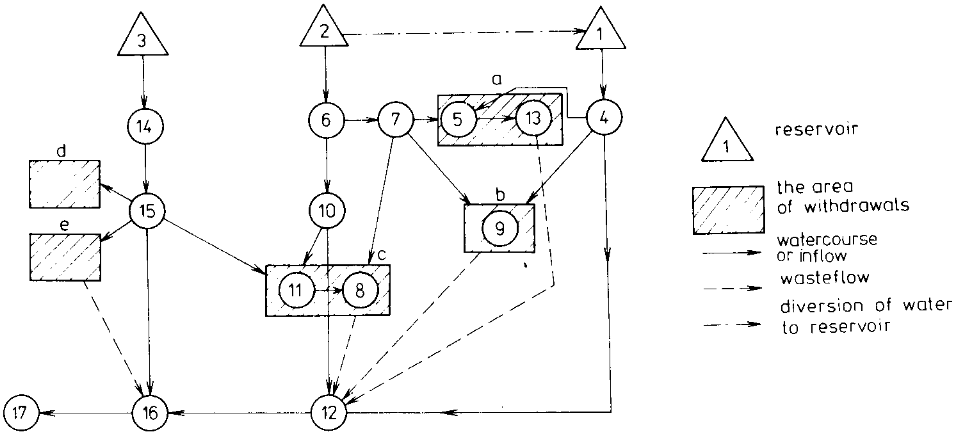

Fig. 10.3 Schematic representation of WRS for the simulation model used in combination with the out-of-kilter algorithm

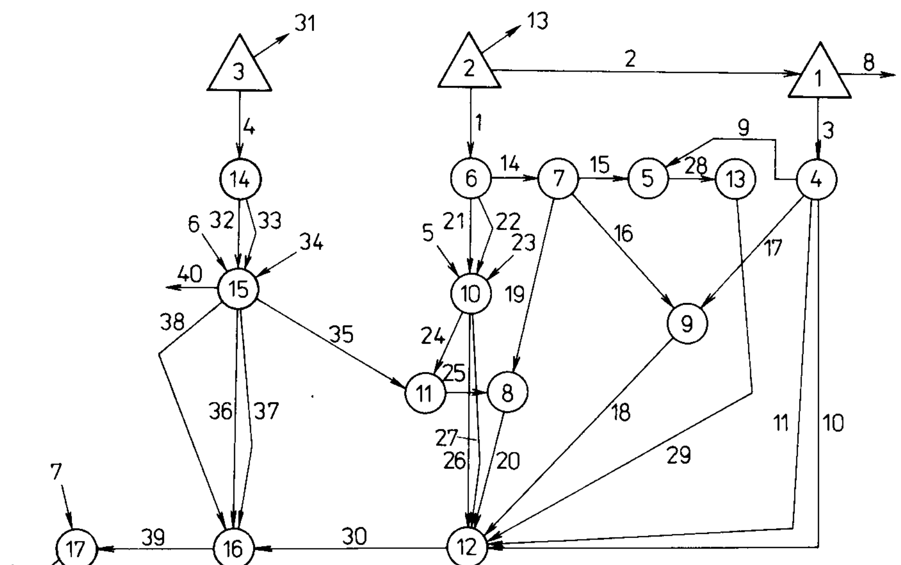

Fig. 10.4 Graph for application of the out-of-kilter method

should be taken into account by connecting these nodes to a hypothetical node called a sink (or super sink) to secure zero circulation, i.e., balance of inflows and outflows in these nodes (in the graph in Fig. 10.4, nodes 13, 8 and 9). This sink node is connected with further nodes by arcs: 8, 13, 31, 40, 41 and 42.

A further hypothetical node called source (or super source) is connected with the graph nodes receiving water. In this way zero circulation is secured in nodes 1, 2, 3, 10, 12, 15 and 17. The source node and sink node are connected by an arc with cost value that is approximately 100 times greater than the other costs. This closes the network and the conditions of the out-of-kilter algorithm are fulfilled (see Fig. 10.2).

The operation of WRS with a defined network is determined by lower and upper boundaries and cost values. If the lower boundary equals the upper boundary (e.g., in some waste flow, in evaporation and other losses of reservoirs), the flow in such an arc is given, and it cannot be changed in the optimization process (e.g., arcs 8, 13, 31, 12, 23, and 34; the values in arcs 8, 13, and 31 are variable for different months; see Table 10.3). The hydrological input of WRS and the active storage of reservoirs 1, 2, and 3 are treated in the same way (arcs 5, 6, and 7).

The water requirements in WRS are given as the upper boundaries of flows in arcs 10, 16, 17, 18, 19, 20, 21, 25, 26, 28, 29, 32, 36, 40 and 42, the upper boundaries of arcs 32, 36, and 42 are variable for different months as shown in Table 10.3.

Table 10.3 Variable requirements in system

No. of arc	Month											
	XI	XII	I	II	III	IV	V	VI	VII	VIII	IX	X
8	120	118	116	118	120	130	116	100	100	100	120	116
13	41	39	38	40	42	47	38	32	33	33	41	39
31	59	58	56	58	60	66	56	49	50	50	59	57
32	190	190	190	190	190	190	190	200	200	200	200	190
36	210	210	210	210	210	210	210	290	290	290	290	210
42	1900	1900	1900	1900	1900	1900	1900	2040	2040	2040	2040	1900

Other values are given in Table 10.2. The values of the lower and upper boundaries for arcs 5, 6, and 7 are not listed in Table 10.2, as they vary in each period. These values for the lower and upper boundaries were used for the simulation run with single-period operation. If multiple-period optimization is carried out, data should be given for each period for each arc.

The output of the out-of-kilter optimization is the allocation of water in investigated period i. In the simulation model, changes due to this optimized allocation are carried out, the model proceeds to step $i + 1$ and the calculation is repeated for the new values of flows and reservoir storage volumes. The main output of this case study is the determination of coefficients in equation (10.25). An example these coefficients is given in Table 10.4 for reservoir 1 for sample months November, February, April and August.

Table 10.4 Regression coefficients in the equation (10.25)

Month	b_0	b_1	b_2	b_3	b_4	b_5	b_6
November	−560	0.299	0.087	0.301	0.966	0.063	−0.041
February	−343	1.083	0	0	1.003	−0.057	0
April	0	0.531	0	0.110	0.967	0	0
August	−730	0.149	0.197	0.165	0.983	0.059	−0.038

The multiple correlation coefficients exceeded 0.98

10.2 THE COMBINATION OF THE CHANCE-CONSTRAINED MODEL AND THE SIMULATION MODEL

Many methods of WRS operation require complicated calculations which make heavy demands on computer time without reflecting the stochastic character of the problem. The chance-constrained model developed by ReVelle *et al.* (1967), ReVelle and Kirby (1970), Joeres *et al.* (1971) and completed by Eastman and ReVelle (1973) by direct capacity determination is an adequate method for relatively simple models using a linear programming technique and the solution of a set of linear equations. The resulting model is stochastic.

10.2.1 The Chance-Constrained Model

The chance-constrained model is first described for a single multi-purpose reservoir. The objective is the minimization of its total storage V_c, satisfying the linear constraints that express goals such as flood control, water supply, minimum storage requirement, minimum and maximum flow constraints. These constraints are expressed in the following ways.

a) Flood control (freeboard) constraint

$$P\{V_{z_i} \leqq V_c - V_{r_i}\} \geqq p_1 \qquad (10.26)$$

The storage of water V_{z_i} in the reservoir shall be less than, or equal to, the total storage reduced by the flood control storage (freeboard) with at least p_1 reliability for period i

b) Minimum storage (pool) constraint

$$P\{V_{z_i} \geqq V_{m_i}\} \geqq p_2 \tag{10.27}$$

The storage V_{z_i} greater than or equal to, minimum V_{m_i} shall be kept with at least p_2 reliability for period i.

c) Minimum release constraint

$$P\{x_i \geqq y_i\} \geqq p_3 \tag{10.28}$$

Release x_i greater than, or equal to, minimum release y_i must be committed with at least p_3 reliability for each time period i.

d) Maximum release constraint

$$P\{x_i \leqq z_i\} \geqq p_4 \tag{10.29}$$

This constraint that release commitment x_i determined for period i shall be less than or equal to the maximum acceptable commitment z_i with reliability p_4, was not used in the calculation. In case studies, it is often done for monthly periods and for flood control in many WRS in ČSSR the monthly release is not an important parameter as maximum flow is reached in shorter time intervals.

10.2.2 Principle and Application of Linear Decision Rule

The linear decision rule assumes that the amount of water x_i, released from a reservoir in a period i is a linear function of the storage of water at the beginning of period i, i.e., at the end of the period $i - 1$ and of the decision parameter b_i, determined by linear programming to optimize some objective function. The release commitment is

$$x_i = V_{z_{i-1}} - b_i \tag{10.30}$$

Substituting the linear decision rule into the continuity (mass balance) equation

$$V_{z_i} = V_{z_{i-1}} + f_i - x_i \tag{10.31}$$

where f_i is the inflow into reservoir in period i, yields

$$V_{z_i} = f_i + b_i \tag{10.32}$$

If the linear decision rule is followed in period $i - 1$, the similar equation for $V_{z_{i-1}}$ is substituted into (10.30), which yields

$$x_i = f_{i-1} + b_{i-1} - b_i \tag{10.33}$$

Using the linear decision rule, the stochastic variable V_{z_i} in chance constraints (10.26)–(10.28) can be substituted by variable f_i. The probability distribution of variable V_{z_i} is unknown and the probability distribution of variable f_i is known or can be calculated. Substituting equations (10.32) and (10.33) into constraints (10.26) to (10.28) and rearranging, yields the constraints in the following form:

$$P\{V_c - b_i - V_{r_i} \geqq f_i\} \geqq p_1 \tag{10.34}$$

$$P\{V_{m_i} - b_i \leqq f_i\} \geqq p_2 \tag{10.35}$$

$$P\{b_i - b_{i-1} + y_i \leqq f_{i-1}\} \geqq p_3 \tag{10.36}$$

Using the cumulative distribution function of inflows f_i, the constraints can be transformed into their deterministic equivalents. From the cumulative distribution function the value $r_i\,(p_1)$ is determined: it is the inverse of this function of f_i evaluated at p_1, or it is the value which the flow in period i exceeds on average in $100\,(1 - p_1)\%$ of time. Similarly, the value $r_i\,(1 - p_2)$ is exceeded on average in $100\,p_2\,\%$ time (and similarly for p_3).

Using these transformations, the following linear program can be formulated:

$$V_c \to \min \tag{10.37}$$

with constraints

$$V_c - b_i - V_{r_i} \geqq r_i(p_1) \tag{10.38}$$

$$V_{m_i} - b_i \leqq r_i(1 - p_2) \tag{10.39}$$

$$b_i - b_{i-1} + y_i \leqq r_{i-1}(1 - p_3) \tag{10.40}$$

The assumption of the effective use of the decision rule in WRS operation requires that the users were able to use the variable release commitments. For many goals, as municipal water supply, the amount of water that exceeds the demand can hardly be utilized. Knowledge of the possibility of water deficits in a future period, however, is important for a reduction of losses due to deficits. The operation policy of reservoirs in practice during drought periods is similar to the linear decision rule, but it is more complicated. However, in many cases the application of the linear decision rule is a better approximation of reality than the operation with a constant release.

Using the linear decision rule separate operational reliability in each month is considered, neglecting the relationship between the probability distributions of flows in adjoining months. The relationship between reliabilities in a chance-constrained model with the linear decision rule and annual occurrence or time-based reliabilities of WRS (see reliabilities R_1 or R_2 in 6.3.7) can hardly be determined generally; they depend on the correlation between monthly flows and other hydrological and operational details. Therefore other methods were sought that could provide relationships, comparable to the overall reliability of WRS operation. More appropriate for this aim is the chance-constrained model with direct release optimization without linear decision rule.

10.2.3 Release Optimization in the Chance-Constrained Model

In the model developed by Curry *et al.* (1973), the releases x_i were directly optimized in period i. The operation of a single multi-purpose reservoir is optimized using the mass balance (continuity) equation and the chance-constraints (10.26) and (10.27), i.e.,

- freeboard constraint,
- minimum storage constraint, with given reliabilities.

The minimum and maximum release constraints are not stochastic, therefore the release must satisfy the minimum and maximum constraint in each case.

In the transformation of the stochastic problem into a deterministic one, the cumulative distribution functions of flows in individual months do not appear, but a cumulation of convoluted distribution functions is necessary (see below). In the continuity equation the evaporation and infiltration losses from the reservoir are added in relation to reservoir storage.

$$V_{z_i} = e_i V_{z_{i-1}} + f_i - x_i \tag{10.41}$$

where e_i is the fraction of water remaining contingent upon losses due to evaporation and infiltration (e.g., for 2% losses in a month i, $e_i = 0.98$). Substituting the equation (10.41) into the chance-constraint (10.26) for $i = 1$ yields

$$P\{e_1 V_{z0} + f_1 - x_1 \leqq V_c - V_{r1}\} \geqq p_1 \tag{10.42}$$

Rearranging and substituting the value $r_1(p_1)$ from the cumulative distribution function such that only $100\,(1 - p_1)\%$ of random values are greater than $r_1\,(p_1)$, the chance-constraint is transformed to

$$V_c - V_{r1} - e_1 V_{z0} + x_1 \geqq r_1(p_1) \tag{10.43}$$

Using equation (10.42) for $i = 2$ and substituting for V_{z1} from equation (10.41) yields

$$P\{e_2 e_1 V_{z0} - (e_2 x_1 + x_2) - V_c + V_{r2} \leqq -(e_2 f_1 + f_2)\} \geqq p_1 \tag{10.44}$$

The *convoluted cumulation distribution function* of the random variables f_1 and f_2 $(e_2 f_1 + f_2)$ evaluated at the point p_1 gives the value $r_2(p_1)$; on the average it will be exceeded in 100 $(1 - p_1)\%$. The chance-constraint is then transformed to the deterministic equivalent:

$$(V_c - V_{r_2} - e_2 e_1 V_{zo}) + (e_2 x_1 + x_2) \geqq r_2(p_1) \tag{10.45}$$

By similar steps the following relationship can be obtained for the n-th period (defining $e_{n+1} = 1$):

$$\left(V_c - V_{r_n} - V_{zo} \prod_{i=1}^{n} e_i\right) + \sum_{i=1}^{n} \left[\left(\prod_{k=i+1}^{n} e_k\right) x_i\right] \geqq r_n(p_1) \tag{10.46}$$

where $r_n(p_1)$ is the value of random variable w_n with probability of exceedance $(1 - p_1)$.

The random variable w_n is defined by

$$w_n = \sum_{i=1}^{n} \left(\prod_{k=i+1}^{n} e_k\right) f_i \tag{10.47}$$

Equation (10.46) is the deterministic equivalent of the chance-constraint (10.26) for the chance-constraint model without the linear decision rule in which the optimal release quantities x_i are the decision variables. In a similar way the deterministic equivalent for the minimum storage chance constraint with a given reliability can be derived. We get

$$V_{m_n} = V_{zo} \prod_{i=1}^{n} e_i + \sum_{i=1}^{n} \left[\left(\prod_{k=i+1}^{n} e_k\right) x_i\right] \leqq r_n(1 - p_2) \tag{10.48}$$

where V_{m_n} is the required minimum storage.

If in operation the withdrawals Q_{d_i}, which are determined for periods i, are additional requirements, then in equations (10.41)–(10.48) instead of x_i the sum $x_i + Q_{d_i}$ is considered. If these water requirements are stochastic, then the value $f_i - Q_{d_i}$ is assumed in all equations instead of flows f_i (Kos *et al.*, 1974).

The model with direct optimization of release x_i can be enlarged for a system of reservoirs interconnected by diversions and pumping of water. The continuity equation for reservoir k in time i is

$$V_{z_{k,i}} = e_{k,i} V_{z_{k,i-1}} + f_{k,i} - x_{k,i} + \sum_{j=1}^{n} (a_{k,j} x_{j,i} - b_{k,j} q_{k,j,i} + b_{j,k} q_{j,k,i}) \tag{10.49}$$

and the optimized variables $q_{k,j,i}$ give the amount of water transferred from reservoir k into reservoir j in time period i. The coefficients $a_{k,j}$, $b_{k,j}$ express the existence of the connection between reservoirs k and j. The coefficients $a_{k,j}$ are equal to one if water is transferred reservoir k to reservoir j; otherwise they are equal to zero. For $k = j$ both the coefficients are equal to zero. The coefficients $a_{k,j}$ are used for water diversion, the coefficients $b_{k,j}$ are used in the same manner for pumping.

Using a special assumption, the solution of equations (10.46) to (10.48) can be simplified (Kos and Zeman, 1976). Assuming $e_i = e$ constant for all periods and assuming the priority of water supply function with constant release $x_i = x$, but with random demands subtracted from inflows, the following simplification is possible

$$\sum_{i=1}^{n}\left[\left(\prod_{k=i+1}^{n} e_k\right)x_i\right] = \frac{1-e^{n-1}}{1-e}\,x = H_n \tag{10.50}$$

$$e^n = B_n \tag{10.51}$$

The inequality (10.46) is reduced to

$$V_c - V_r - V_{z0}B_n + H_n x = \max_j \left[r_n(p_1)\right] \tag{10.52}$$

and the inequality (10.48) is changed to

$$V_{m_n} - V_{z_0}B_n + H_n x = \min_j \left[r_n(1-p_2)\right] \tag{10.53}$$

In inequalities (10.52) and (10.53) the j-th month is the first month of convolution. For the value $V_{m_n} = 0$ this simplification yields

$$V_z = V_{z_0} = \max_n \left[\frac{1}{B_n}\left(H_n x - \min_j \left(r_n(1-p_2)\right)\right)\right] \tag{10.54}$$

and the total storage of the reservoir is

$$V_c = V_{c_n} = V_r + B_n V_z - H_n x + \max_j \left[r_n(p_1)\right] \tag{10.55}$$

In equation (10.55) the value n is used, which yields a maximum in equation (10.54).

The solution requires the knowledge of the cumulative convoluted distribution function of flows. As theoretical probability distribution function for this purpose the normal distribution, log-normal distribution, and Gumbel's distribution were tested. The parameters were derived by moments and by quantiles. The best fit of empirical and theoretical data and the easiest use on a computer was achieved by *Gumbel's distribution with method of quantiles* (Dupačová and Kos, 1979) (however, the statistical tests of goodness-of-fit were also satisfied by other distributions).

Using the method of quantiles, values such as $r_n(p_2)$ were determined. The quantiles for 5% ($X1$) and 95% ($X2$) were determined from empirical distributions. Quantities A, S, and Z were calculated, using these values:

$$A = \ln\left(-\ln\left(1-p_2\right)\right) \tag{10.56}$$

$$S = \frac{4.067}{X1 - X2} \tag{10.57}$$

$$Z = 0.27X1 + 0.73X2 \tag{10.58}$$

Then, $r_n(p_2)$ could be calculated

$$r_n(p_2) = \frac{A}{S} + Z \tag{10.59}$$

The calculational and economic constraints limited the maximum values of n to 48, i.e., 4 years.

10.2.4 Chance-Constrained Model and Simulation Model

In the General Water Plan of Czechoslovakia the deterministic simulation model was used in most case studies. In some subsystems of WRS, when a relatively high reliability was required for municipal or industrial water supply with high response to eventual water deficits, the stochastic model was used. The combination of the chance-constrained model with the simulation model was one of the applied methods of stochastic models. The main aim of this combination was the determination of reliability of withdrawals (Kos, 1975).

Table 10.5 Hydrological characteristics of points in WRS

No. of point	1	2	3	4	5	6	7	8	9
Mean annual flow in $m^3\ s^{-1}$	0.96	5.56	3.18	1.45	1.45	3.18	5.56	0.96	21.7

Let us consider a system whose schematic representation is in Fig. 10.5. The goal of four reservoirs (Nos. 1 to 4) is the water supply in points 5 and 9. The minimum flows in the rivers at these points are 0.32 and 1.72 $m^3\ s^{-1}$, respectively, and the water requirements for consumption (trans-basin diversion) are 0.92 and 7.21 $m^3\ s^{-1}$, respectively. In addition, the amount of possible withdrawals without consumption at points 6 to 8 is investigated.

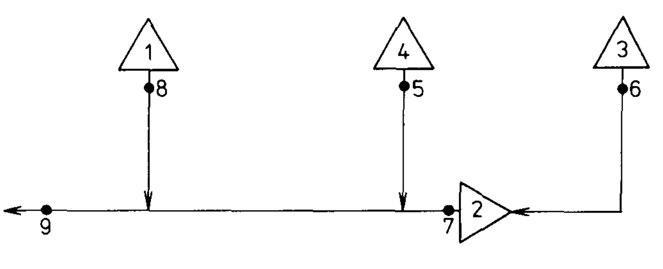

Fig. 10.5 Schematic representation of WRS when the chance-constrained model was applied

The determinstic simulation model was used for the evaluation of variable reservoir releases for river flow regulation at point 9 with different alternatives of operation policy. As the main results of simulation runs with the necessary reservoir capacities, the operation without deficits using the monthly flows in the period of 1931 – 1970 was determined. Then the chance-constrained model was used for the same WRS using the above simplification (e_i = e; x_i = x const) with direct release determination and stochastic release requirements caused in reservoirs due to river flow regulation and given as the output of the deterministic simulation model.

Table 10.6 The basic input parameters of WRS

No. of reservoir	1	2	3	4
Active storage used in computation (10^6 m^3)	30.5	45.0	68.0	54.6
Minimum release (m^3 s^{-1})	0.12	1.00	0.46	0.32

Table 10.7 Main output values of WRS

No. of alternative	No. of point	Release interval n	Active storage [10^6 m^3]	Draft [m^3 s^{-1}]
1	8	33	27.50	0.89
2	8	48	72.32	1.07
3	7	7	23.74	3.00
4	7	20	39.52	3.60
5	7	20	84.73	4.32
6	6	16	49.92	1.00
7	6	16	59.41	1.20
8	6	27	70.92	1.44
9	5	44	64.96	1.27
10	5	42	52.23	1.17

The hydrologic time series at points of the WRS are characterized by the main parameter – the annual mean flows given in Table 10.5. The active storage of reservoirs used in computation and minimum reservoir releases used in the simulation model are given in Table 10.6. The results of the chance-constrained model, i.e., the relationship of draft Q_n and the necessary active storage V_z for the reservoir operation for river flow regulation at points 5 and 9, for 97 % reliability, and the optimal length n of the convolution period used in the equations (10.54) and (10.55) are given in Table 10.7.

By logarithmic interpolation, i.e., using the relation a, b – constants

$$Q_n = a + b \log (V_z) \tag{10.60}$$

Table 10.8 The active storage volumes of reservoirs of WRS for reliability of release 97%

No. of reservoir	1	2	3	4
Active storage volume (10^6 m^3)	32	50	75	61
No. of point	8	7	6	5
Draft (m^3 s^{-1})	0.92	3.96	1.50	1.23

which can be used approximately for the whole range of required release and that fits better than the linear interpolation, the values of the draft in the points of WRS for the given active storage of reservoirs were determined. The active storages V_z are greater than those used for calculations in the simulation model, as they were not reduced by losses. These losses were included in the chance-constrained model in such a manner that the losses in both models were approximately the same[1]).

Table 10.8 indicates that the draft at point 5 (i.e., 0.32 + 0.92 = 1.24 m^3 s^{-1}), which was derived by the deterministic simulation model, has a 97 % reliability as derived by the chance-constrained model because in both models approximately the same values of draft appeared. If the 99 % reliability were required, differences would occur and a higher active storage would be necessary for draft 1.24 m^3 s^{-1}.

10.2.5 Conclusions of the Chance-Constrained Model

Simulation modelling is a very good technique, if the validity of the model is measured by its ability to approximate reality correctly. However, optimization is not implicitly included in the simulation model and the optimum is reached by many

[1]) Assuming 2% monthly losses of reservoir storage the total loss for 9 months of the drought period will be $0.98^9 = 0.83$ and for 12 months $0.98^{12} = 0.79$, which is approximately 17–21%.

repetitions of calculation runs changing the input parameters. If the stochastic character of the modelled phenomena is to be reflected, i.e., the stochastic simulation model is required, then sophisticated methods are necessary for the generation of hydrologic input data. As optimization of the stochastic simulation model is time-consuming, it is only suitable for the detailed investigation. Therefore, in planning the verification of results of the deterministic simulation model is often done in only few cases.

For these reasons some other methods were tested to reflect the stochastic character in the models. A combination of the deterministic simulation model with the chance-constrained model is an appropriate method for this goal, which can be used for the determination of reliability of a WRS and for its optimization with relatively low computer time requirements.

The use of the linear decision rule was tested. However, the results of the chance-constrained model can hardly be compared with the results of the simulation model with fixed target demands on WRS, i.e., with demands predetermined before the computation (possibly variable during the year). Such fixed demands were typical for municipal and industrial water supply. Water requirements for irrigation are random variables, dependent on meteorological, agricultural and other elements, but related in a small degree to the water commitment as in the linear decision rule. Partly, the linear decision rule can be used for a combination of flood control and water power plant operation where the water commitment could be used for dummy power production.

The direct release optimization without the linear decision rule was more advantageous for the combination of the simulation model with the chance-constraint model. The results can be easily compared with those of the simulation model with occurrence-based reliability.

The chance-constrained model with the linear decision rule has a direct solution or a linear program of a small dimension can be used. If the direct release optimization (without simplification as shown) is used then for longer periods (e.g., for 20 years) extensive linear programs are obtained. For example, for 15 reservoirs with one diversion each, over 20 years, with monthly periods, the number of constraints (without non-negativity constraints) would be 7200 and the number of bounded variables 7200. If water supply is a priority goal, the investigated period can be reduced to 60 months or the described simplification can be used. This simplification is advantageous for withdrawals from reservoirs for the municipal water supply with priority over river flow regulation for the water supply for industry and irrigation.

The operation policy of WRS with priorities does not often yield the overall optimum and the possibilities of the chance-constrained model are investigated especially in combination with the simulation model to utilize this combination for better response to minimum storage and free-board constraints (Kos, 1079a). However, the present application of this combination, which was used for operation reliability determination under conditions given by the simulation model, is a progressive technique for many cases.

10.3 STOCHASTIC ANALYTICAL MODEL

The analytical model has to meet a number of requirements to reflect the modelled reality correctly. It should include the longterm (over-year; carry over) and short-term (seasonal; within year) release operation, storage requirements, and flood control function of reservoirs, optimization of hydroelectric power plant capacities, targets for water supply and for various goals, low flow augmentation, etc. In addition, the analytical model must contain some economic functional relationship as the dependence of capital, operation, maintenance and repair costs on reservoir storage, relationship between reservoir water level and reservoir storage, the water supply gross benefits from hydroelectric power and gross recreational benefits, and loss functions expressing the reduction of benefits due to the deficits in water supply, reduction of recreation possibilities and hydroelectric power generation below the level of firm energy commitment.

If a situation of water resources development is modelled, the model should have dynamic properties.

These conditions can be met by a simulation model, however, this model does not include the system optimization. At present there is no analytical model that can satisfy all the conditions listed. Some requirements have to be weakened and some have to be omitted. The character of the task implies the modelling of the stochastic character of the process. The main function of WRS does not appear during periods with average flows, but during extreme deviations such as drought or flood situations. Therefore a stochastic optimization model that includes the computation of the risk of system failure is necessary. Such a model was developed by Jacoby and Loucks (1972) and is called the *screening model.*

The principle of the model is the probabilistic evaluation of different stages of reservoir storage, uncontrolled and controlled flows, reservoir releases, maximization of the expected benefits depending on these probabilities using the linear programming technique. In this model the discrete values of flows and reservoir storage are used with the corresponding probability of occurrence that are used for the determination of active and flood control storage of reservoirs, parameters of the hydroelectric power plants and parameter for operation of the system.

The main goals are represented by water supply (V), hydroelectric power (H), flood control (O), and recreation (R), respectively. The site or location (point) in the basin is denoted by the subscript g. For example the annual gross flood control benefits are U_g^{O}. The total annual costs at site g are N_g. The objective function of the model, which is maximized, is formed by the expected (E is the symbol of expectation) total annual net benefits from all sites, i.e.

$$\max E \sum_{g} (U_g^{\mathrm{V}} + U_g^{\mathrm{H}} + U_g^{\mathrm{O}} + U_g^{\mathrm{R}} - N_g) \tag{10.61}$$

In the simulation model the basic time period is one month. In the analytical model the number of intervals has to be reduced. In the given example the year was divided into two intervals: the spring period with high flows (March, April, May) and the rest of the year.

Table 10.9 Conditional and unconditional probabilities at control point Vilémov on the Jizera River

Conditional probabilities $P_{i,j}^t$ for $t = 1$			
	j		
i	1	2	3
1	0.533	0.592	0.452
2	0.167	0.176	0.170
3	0.300	0.232	0.378

Conditional probabilities $P_{i,j}^t$ for $t = 2$			
	j		
i	1	2	3
1	0.605	0.616	0.310
2	0.184	0.151	0.304
3	0.211	0.233	0.386

Unconditional probabilities $P_{i,t}$			
	i		
t	1	2	3
1	0.517	0.215	0.268
2	0.524	0.170	0.306

In each of these time intervals some streamflow intervals are determined. In the example discussed three intervals (states) were used, describing low, mean and high streamflows. The range of possible flows is divided into three consecutive intervals, i.e. i or j (i.e., $i = 1, 2$, or 3; $j = 1, 2$, or 3), in such a way that for each period t the conditional probabilities $p^t_{i,j}$ are the same for all sites. This conditional probability expresses the probability that the unregulated flow will be in interval j in period $t + 1$, given an unregulated flow in interval i in period t. The choice of the same probabilities simplifies the formulation of the linear program.

The unconditional steady state probabilities $P_{i,t}$ expressing the probability of each flow interval i in each period t, are calculated from conditional probabilities $p^t_{i,j}$ by solving the following set of simultaneous linear equations

$$P_{j,t+1} = \sum_{i=1}^{n} P_{i,t} P^t_{i,j} \qquad \text{for } j = 1, 2, \ldots, n - 1 \quad t = 1, \ldots, T \tag{10.62}$$

$$\sum_i P_{i,t} = 1 \qquad \text{for } t = 1, \ldots, T \tag{10.63}$$

In this example, $n = 3$ and $T = 2$. For the values of conditional probabilities given in the Table 10.9 for the gauging station Vilémov, the following set of equations will be obtained

$$\begin{aligned} P_{1,2} &= P_{1,1} \cdot 0.533 + P_{2,1} \cdot 0.592 + P_{3,1} \cdot 0.452 \\ P_{2,2} &= P_{1,1} \cdot 0.167 + P_{2,1} \cdot 0.176 + P_{3,1} \cdot 0.170 \\ 1 &= P_{1,1} + P_{2,1} + P_{3,1} \\ P_{1,1} &= P_{1,2} \cdot 0.605 + P_{2,2} \cdot 0.616 + P_{3,2} \cdot 0.310 \\ P_{2,1} &= P_{1,2} \cdot 0.184 + P_{2,2} \cdot 0.151 + P_{3,2} \cdot 0.304 \\ 1 &= P_{1,2} + P_{2,2} + P_{3,2} \end{aligned} \tag{10.64}$$

By solving these equations the values of the unconditional probabilities in Table 10.9 are obtained. In equations (10.62) to (10.64) a cycle was assumed; then the period $t = 1$ is followed by $t = 2$ and this period is followed again by $t = 1$, etc.

In (10.64), the equation for $j = 3$ cannot be used because then the sum of the first equations (i.e., for $t = 1$ and $j = 1, 2, 3$) would equal the fourth equation (expressing

Table 10.10 Discrete values of uncontrolled flows $G_{i,t,1}$ (10^6 m^3)

	i		
t	1	2	3
1	41.52	70.19	83.00
2	53.25	88.76	108.48

that the sum of probabilities $P_{i,t}$ is equal to one for $i = 1, 2, 3$). The sum of conditional probabilities $P^t_{i,j}$ for the given i, t and $j = 1, 2, 3$ is equal to one and the equations would be linearly dependent.

In further calculation the streamflow values are replaced in each interval i or j by one discrete value $G_{i,t,h}$, where h denotes the gauging station, t is the time period, in such a way as to maintain the first two moments of the probability distribution of uncontrolled flows. The third condition is maintaining $G_{2,t,h}$ in the middle of the medium interval for $i = 2$. The values of $G_{i,t,h}$ for one station are given in Table 10.10. The values $I_{i,t,g}$, the discrete uncontrolled flows at each site g within the basin, are computed from values $G_{i,t,h}$ by reduction coefficients $k_{h,g}$:

$$I_{i,t,g} = k_{h,g} G_{i,t,h} \tag{10.65}$$

$$k_{h,g} = \frac{Q^g_a}{Q^h_a} \tag{10.66}$$

where Q^g_a is the mean annual flow in site g and Q^h_a is the mean annual flow in site h (gauging station).

The flows controlled by a subsystem of WRS $Q_{i,t,g}$ in interval i, in period t and site g are determined by the mass balance equation (10.67) as the uncontrolled flows reduced by the change of flows immediately upstream from site g and consumption

$$Q_{i,t,g} = I_{i,t,g} - \sum_A (I_{i,t,a} - R_{i,t,a}) - \sum_B Z_{i,t,b} \quad \text{for all } i, t \tag{10.67}$$

where $a \in A$ (A is the set of all control sites immediately upstream from site g) and $b \in B$ (B is the set of all withdrawals sites between those control sites and site g), $Z_{i,t,b}$ is the consumption in sites b, $R_{i,t,a}$ is the expected release in sites a; both in period t and interval i. All values in equation (10.65) and following equations are given in mil. m^3 in period t.

In equation (10.67), the occurrence of all values is assumed within the same state interval i. It is a simplifying assumption, which is realistic in flow regimes with similar time patterns. For this analytical model this assumption can be used for all rivers in Czechoslovakia, with the exception of the Danube.

In further investigations, the reservoir storage volumes are divided into three discrete state intervals denoted as k or m (i.e., $k = 1, 2$, or 3; $m = 1, 2$, or 3). Then states k or m correspond to intervals just as the states i or j designate the intervals of streamflows. However, the storage volume intervals k and m are unknown, therefore the reservoir storage volumes $S_{k,t,g}$ or $S_{m,t+1,g}$ are also unknown. The reservoir releases depend on these values and they can therefore be denoted as $R_{k,i,m,t,g}$; using the mass balance equation we get

$$R_{k,i,m,t,g} = S_{k,t,g} + Q_{i,t,g} - S_{m,t+1,g} \tag{10.68}$$

for all k, i, m, t.

To secure a uniform filling and releasing of reservoirs in the whole WRS and to satisfy the assumption in equation (10.67) the relationship for the determination of state m is the following[1]):

$$m = \text{entier} \frac{k + i}{2} \tag{10.69}$$

As m is uniquely defined by k, i, the subscript m can be omitted.

The reservoir release depends on the reservoir state k and therefore it is denoted as $R_{k,i,t,g}$; similarly the controlled flow $Q_{k,i,t,g}$ and consumption $Z_{k,i,t,g}$ depend on state k and the subscript k is used. Therefore the equation (10.67) becomes (10.70)

$$Q_{k,i,t,g} = I_{i,t,g} - \sum_A (I_{i,t,a} - R_{k,i,t,a}) - \sum_B Z_{k,i,t,b} \tag{10.70}$$

for all k, i, t.

The change of the equation (10.68) consists in the omission of the subscript m in release R.

The conditional probability for different states of flows and reservoir storage volumes $P_{k,i,m,j,t}$ is the probability of having an initial storage volume within interval m and a flow within interval j in period $t + 1$ given an initial storage volume within interval k and a flow within interval i in period t. Since the conditional probabilities $P^t_{i,j}$ are the same at each reservoir site and the relationship of m to k, i is given by the equation (10.69), the conditional probabilities are:

$$\begin{aligned} P_{k,i,m,j,t} &= P^t_{i,j} \text{ if } m = \text{entier} \frac{k + 1}{2} \\ P_{k,i,m,j,t} &= 0 \quad \text{otherwise} \end{aligned} \tag{10.71}$$

Knowing these conditional probabilities the steady state joint probabilities $P'_{k,i,t}$ can be calculated in the same way as in the equation (10.62):

$$\begin{aligned} P'_{m,j,t+1} &= \sum_k \sum_i P'_{k,i,t} P_{k,i,m,j,t} \\ &\text{for } j = 1, 2, \ldots, n - 1 \\ &\quad\;\; t = 1, 2, \ldots, T \\ &\quad\;\; m = 1, 2, \ldots, M \end{aligned} \tag{10.72}$$

$$\sum_k \sum_i P'_{k,i,t} = 1 \quad \text{for } t = 1, 2, \ldots, T \tag{10.73}$$

[1]) The function entier gives the maximum integer portion of the argument, i.e., of $(k + i)/2$, e.g., entier $(2.5) = 2$.

For the site Vilémov on the Jizera River the probabilities $P'_{k,i,t}$ are given in Table 10.11. The probabilities $P'_{k,i,t}$ are not determined by the choice of intervals i, j, k and m uniquely. The reservoir storage volumes $S_{k,t,g}$ and the release $R_{k,i,t,g}$ are

Table 10.11 Unconditional probabilities $P'_{k,i,t}$ in site Vilémov on the Jizera River

In period $t = 1$			
	k		
i	1	2	3
1	0.382	0.112	0.135
2	0.134	0.103	0.133
3	0	0	0.001

In period $t = 2$			
	k		
i	1	2	3
1	0.342	0.106	0.181
2	0.182	0.063	0.125
3	0	0	0.001

influenced by the joint probabilities $P'_{k,i,t}$, but not determined by them. Therefore a set of constraints can be used and the solution optimized. These constraints are given by the goals of the system, e.g., by water supply, hydroelectric power, flood control, recreation, etc.:

a) The water supply constraints consider the annual water supply targets T_g^V that are allocated to period t and site g using the variables $a_{g,t}$. If the target is not met, deficits $D^V_{k,i,t,g}$ occur. The total water supply available in site g in period t is limited by $Q_{k,j,t,g}$. Then

$$a_{g,t}T_g^V - D^V_{k,i,t,g} \leqq Q_{k,i,t,g} \quad \text{for all } k, i, t \tag{10.74}$$

The consumption $Z_{k,i,t,g}$ in site g is assumed to be a portion $c_{g,t}$ of the actual withdrawal, i.e.

$$Z_{k,i,t,g} = c_{g,t}(a_{g,t}\, T_g^V - D^V_{k,i,t,g}) \tag{10.75}$$

for all i, t.

If the benefits function associated with the water supply target is f_g^{V} and the loss function is $F_{t,g}^{\mathrm{V}}$, then the annual expected water supply benefits are

$$E(U_g^{\mathrm{V}}) = f_g^{\mathrm{V}}(T_g^{\mathrm{V}}) - \sum_t \sum_k \sum_i P'_{k,i,t} F_{t,g}^{\mathrm{V}}(D_{k,i,t,g}^{\mathrm{V}}, a_{g,t} T_g^{\mathrm{V}}) \tag{10.76}$$

b) The production of hydroelectric power at each reservoir site is dependent on a number of factors: the installed generating capacity of the power plant H_g, the flow through turbines, the mean storage head $H_{k,m,t,g}$, which depends on the initial storage volumes in each period, the number of hours in each period h_t, and the plant factor K_t indicating the position of the hydropower production in period t in the electric grid. The total annual firm power target T_g^{H} is apportioned to each period by the fraction coefficient $b_{t,g}$. Then the firm energy requirement in site g and period t will be $h_t K_t b_{t,g} T_g^{\mathrm{H}}$. If this requirement is not met, deficits $D_{k,i,t,g}^{\mathrm{H}}$ may occur; otherwise "dump" energy $E_{k,i,t,g}^{\mathrm{H}}$ may be generated. If k_g denotes the constant for converting the product of flow, head and plant efficiency into energy, then the reservoir release $R_{k,i,t,g}$ provides the following maximum of energy produced:

$$h_t K_t b_{t,g} T_g^{\mathrm{H}} - D_{k,i,t,g}^{\mathrm{H}} + E_{k,i,t,g}^{\mathrm{H}} \leqq k_g R_{k,i,t,g} H_{k,m,t,g} \tag{10.77}$$

for all k, i, t and m = entier $[(k+1)/2]$. The total energy is constrained by the installed generating capacity H_g. Then

$$h_t K_t b_{t,g} T_g^{\mathrm{H}} + E_{k,i,t,g}^{\mathrm{H}} \leqq h_t H_g \tag{10.78}$$

for all k, i, t.

The energy produced has to be positive, i.e., the deficit cannot surpass the requirements

$$h_t K_t b_{t,g} T_g^{\mathrm{H}} - D_{k,i,t,g}^{\mathrm{H}} \geqq 0 \tag{10.79}$$

If function $f_{g,t}^{\mathrm{H}}$ expresses the benefit function for hydroelectric power and the loss function is $F_{g,t}^{\mathrm{H}}$, including not only the deficits of the firm energy, but also the dump energy (as a negative loss), then the expected annual power benefits are

$$E(U_g^{\mathrm{H}}) = \sum_t f_{g,t}^{\mathrm{H}}(b_{t,g} T_g^{\mathrm{H}}) - \sum_t \sum_k \sum_i P'_{k,i,t} F_{g,t}^{\mathrm{H}}(D_{k,i,t,g}^{\mathrm{H}}, E_{k,i,t,g}^{\mathrm{H}}, T_g^{\mathrm{H}}) \tag{10.80}$$

c) The reservoir storage volumes A_g are determined in existing reservoirs; for other reservoirs the following constraint must be satisfied

$$A_{\min,g} \leqq A_g \leqq A_{\max,g} \tag{10.81}$$

The maximum is determined by hydrological, morphological and economical conditions. A minimum value $A_{\min,g}$ is necessary for the reservoir function to be important for WRS. The sum of the maximum active storage $V_{t,g}$ and of the flood control storage in period t, $V_{t,g}^{\mathrm{O}}$ must be less than, or equal to, the total storage volume A_g, i.e.

$$\max_t V_{t,g} + \max_t V_{t,p}^{\mathrm{O}} \leqq A_g \quad \text{for all } t. \tag{10.82}$$

Assuming the relationship in the flood control storage volume in WRS, the flood damage reduction cannot be expressed directly as a function of the values $V^{O}_{t,g}$, but as a function of some equivalent values $V^{e}_{t,g}$ that are expressed as the actual flood storage capacity $V^{O}_{t,g}$ plus the values $V^{h}_{t,g}$ that include the effect of the upstream reservoir on reducing the peak of a standard project flood, then

$$V^{e}_{t,g} = V^{O}_{t,g} + V^{h}_{t,g} \tag{10.83}$$

and the annual expected flood control benefits are:

$$E(U^{O}_{g}) = \sum_{t} f^{O}_{g,t}(V^{e}_{t,g}) \tag{10.84}$$

where the benefit function $f^{O}_{g,t}$ was used.

d) The recreation benefits can be assumed to depend on the stability of the reservoir pool and storage volume $V_{k,t,g}$. These volumes are partitioned into the seasonal target storage volume T^{R}_{g}, deficit $D^{R}_{k,t,g}$ and excess $E^{R}_{k,t,g}$, then

$$V_{k,t,g} = T^{R}_{g} - D^{R}_{k,t,g} + E^{R}_{k,t,g} \tag{10.85}$$

The number of visitors may vary at different periods of the year. This cyclic variation is expressed by coefficients $r_{g,t}$, the sum of which for all periods t in each site g is equal to one. Using the value of one visitor day and the number of expected visitors, the recreation benefit function f^{R}_{g} and the recreation loss function $F^{R}_{t,g}$ can be derived. The annual expected recreation benefits are

$$E(U^{R}_{g}) = f^{R}_{g}(T^{R}_{g}) - \sum_{t}\sum_{k}\sum_{i} P'_{k,i,t} r_{g,t} F^{R}_{t,g}(D^{R}_{k,t,g}, E^{R}_{k,t,g}, T^{R}_{g}) \tag{10.86}$$

In order to enable the use of the linear programming technique all benefit and loss functions are piecewise linear.

e) The computation of expected net benefits requires the values of annual costs in equation (10.61) associated with the construction, the operation, the maintenance and the repair of reservoirs, power plants and recreation facilities. It is assumed that all costs are functions of capital costs and that these costs are defined for the total reservoir storage capacity A_{g}, the power plant capacity H_{g} and recreation facilities T^{R}_{g}. Thus all the components of the objective function (10.61) have been defined. The optimal alternative is then calculated using a linear program under the given constraints. It is convenient to use such a program where the boundaries of the variables are included in a special boundary section. The problem is simplified and the number of constraints is reduced. Using a heuristic approach many constraints can be omitted as they are satisfied by the fulfilment of other conditions.

10.4 THE DESCRIPTION OF SIMPLIFIED OUT-OF-KILTER ALGORITHM APPLICATION IN AN EXAMPLE

In the description of the algorithm, the following notation us used: $c_{i,j}$ is the cost value in arc (i, j), $h_{i,j}$ is the upper boundary of flow in arc (i, j), y_i is the circulation in node i, and $x_{i,j}$ is the flow along arc (i, j).

The circulation y_i in node i is determined by

$$y_i = \sum_i x_{i,j} - \sum_j x_{j,i} \tag{10.87}$$

The corresponding values that change during calculation are denoted by capital letters (e.g., $C_{i,j}$ correspond to $c_{i,j}$). In this simplified algorithm the lower boundary of the arc flow is assumed to be zero.

The algorithm includes the following five steps:

1. The initial values are $C_{i,j} = c_{i,j}$; $H_{i,j} = h_{i,j}$; $X_{i,j} = 0$; $Y_i = y_i$,

2. The values $c_{i,j}$ and $C_{i,j}$ are transformed to non-negative values. *If* $C_{i,j} \leqq 0$ then a further transformation is carried out for each node-source (i.e., where $y_i > 0$) all the values $C_{i,j}$ are reduced by the quantity $\min_j C_{i,j}$ and all the values $C_{j,i}$ are increased by the same quantity.

3. The maximum flow is calculated: starting in a node-source, i.e., where $Y_i > 0$, the flow is allocated to arcs where $C_{i,j} = 0$ labelling the nodes of the graph in the following manner: all nodes i, where $Y_i > 0$ are labelled by a couple (N_i, T_i) defining $N_i = 0, T_i = Y_i$.

In addition, the nodes to which the flow can be directed are also labelled. Assume that node k has been labelled. Then all the unlabelled nodes j are labelled by numbers $N_j = -k$ and $T_j = \min(T_k, X_{j,k})$ if $X_{j,k} > 0$ and $C_{j,k} = 0$. Then the nodes will be labelled (using node m as the starting point) by the numbers $N_m = k$; $T_m = \min(T_k, H_{k,m})$ for all unlabelled nodes m where $C_{k,m} = 0$, $H_{k,m} > 0$.

Having treated node k in this way, the following labelled node is taken and using this labelling procedure the adjoining nodes are labelled. The labelling procedure proceeds till some node with $Y_n < 0$ is reached or as long as it is possible. In the former case flow n can be increased, therefore, go to step 4. In the latter case it is not possible, therefore go to step 5 for potential improvement.

4. Partition of flows: Assume $T = \min(T_n, |Y_n|)$; add T to Y_n and $X_{i1,n}$ where $i1 = N_n$ and subtract from $H_{i1,n}$. Go to node $i1$, if $N_{i1} < 0$; subtract T from $X_{i1,i2}$, $i2 = |N_{i1}|$ and add to $H_{i1,i2}$. If $N_{i1} > 0$ then add T to $X_{i2,i1}$ and subtract from $H_{i2,i1}$. Go to node $i2$ and repeat this procedure until node ik is reached where $N_{ik} = 0$. Subtract T from Y_{ik}. If $\sum Y_i = 0$ for all i there is $Y_i \geqq 0$, then the flow is optimal.

5. Change the potentials: assume A is the set of labelled nodes. For each i that is the element of A $(i \in A)$, the value F_i is determined.

$$F_i = \min\left(\min_{j \in W_i} C_{i,j}, \min_{j \in U_i} |C_{j,i}|\right) \tag{10.88}$$

where W_i is the set of nodes where for all j that are not elements of A are $C_{i,j} > 0$, and U_i is the set of nodes where for all j that are not elements of A are $C_{j,i} < 0$.

Then all values $C_{i,j}$ are reduced and $C_{j,i}$ are increased for each $i \in A$ by the value $F = \min F_i$. Go to step 3.

The correctness of this algorithm can be proved by comparison with the equations and procedures described in section 10.1. It is apparent that the algorithm uses

$$\begin{aligned} C_{i,j} &\geqq 0 \quad \text{for } X_{i,j} = 0 \\ C_{i,j} &= 0 \quad \text{for } 0 < X_{i,j} < h_{i,j} \\ C_{i,j} &\leqq 0 \quad \text{for } X_{i,j} = h_{i,j} \end{aligned} \tag{10.89}$$

$C_{i,j} = c_{i,j} - Z_i + Z_j$ where Z_i and Z_j are the final quantities of value F that decreases (Z_i) or increases (Z_j) the cost $c_{i,j}$. By comparing it with the out-of-kilter numbers, assuming $d_{i,j} = 0$ (the lower boundary is zero and $C_{i,j} = f_{i,j}$ in equation (10.24)) the corectness of the described algorithm and its convergence to optimal flow is clear.

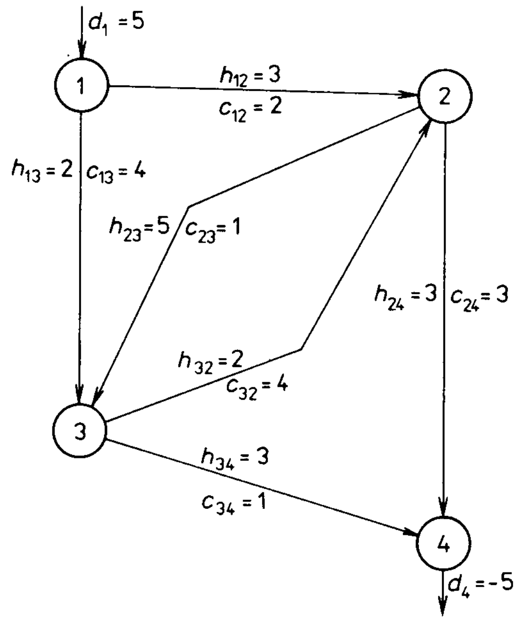

Fig. 10.6 An example of the algorithm out-of-kilter application

The method described is illustrated in a numerical example. The optimal flow in the network in Fig. 10.6 is sought. In step 1 we have $X_{i,j} = 0$. In step 2 the costs of arcs starting in source are changed and we have $C_{1,2} = 0$, $C_{1,3} = 2$. In step 3 the nodes $i = 1, 2$ are labelled by couples $N_1 = 0$; $T_1 = 5$; $N_2 = 1$; $T_2 = 3$. Further labelling is not possible, go to step 5. We have $F = 1$. The costs in arcs are changed: $C_{1,2} = 0$, $C_{1,3} = 1$, $C_{2,3} = 0$, $C_{3,2} = 5$, $C_{2,4} = 2$, $C_{3,4} = 1$[1]).

[1]) $C_{1,2}$ does not change, as it is increased and decreased by the same value ($F = 1$), as $1 \in A$ and $2 \in A$.

In step 3 let us label node $i = 3$; $N_3 = 2$, $T_3 = 3$, go to step 5. We have $F = 1$. If the costs in arcs are changed, then $C_{1,2} = 0$, $C_{1,3} = 1$, $C_{2,3} = 0$, $C_{3,2} = 5$, $C_{2,4} = 1$, $C_{3,4} = 0$, node $i = 4$ can be labelled $N_4 = 3$, $T_4 = 3$, go to step 4. The initial zero flow is increased by $T = \min(3,5) = 3$, then we have $X_{3,4} = 3$, $X_{2,3} = 3$, $X_{1,2} = 3$ and the values $H_{i,j}$ are changed: $H_{3,4} = 0$, $H_{2,3} = 2$, $H_{1,2} = 0$. Then it is possible to label node $i = 1$ as $N_1 = 0$, $T_1 = 2$. Further $F = 1$. The nodes $i = 3,2$ are labelled $N_3 = 1$, $T_3 = 2$, $N_2 = -3$, $T_2 = 2$. Again $F = 1$. If node $i = 4$ is labelled $N_4 = 2$, $T_4 = 2$, the flow can be changed by 2 and the optimal flow will result: $X_{1,2} = 3$, $X_{1,3} = 2$, $X_{2,3} = 1$, $X_{3,4} = 3$, $X_{2,4} = 2$.